Erratum: Focus wave modes in conducting media\(^1\)

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(Ref.: Can. J. Phys. 72, 293 (1994))

A family of localized solutions of the homogeneous equation for waves in lossy media is derived. These solutions are analogues to the set of Brittingham’s focus wave modes (J. Appl. Phys. 54, 1179 (1988)).


[Traduit par la rédaction]


1. Introduction

In recent years many novel nonseparable solutions of the homogeneous wave equation and related equations have been obtained, among which Brittingham’s focus wave modes (FWM) [1, 2] and other localized solutions [3–5] received special recognition. Using (i) substitutions and (or) transforms that reduce the original equations to those having known solutions or (ii) full (space-time) Fourier transforms, the results were extended to the homogeneous Klein–Gordon equation [6–10] and the equation for waves in lossy media [7, 8].

Our reasons for writing this paper are as follows.

(i) We derive a set of localized waves in lossy media that are full analogues to the set of Brittingham’s FWM [2] (in the sense that the former turns into the latter when the conductivity tends to zero).

(ii) We call attention to another approach for constructing solutions of partial-differential equations. Being based on a natural method for deriving solutions in cylindrical coordinates, this straightforward approach turns out to be extremely efficient both for homogeneous and inhomogeneous [11] wave equations. The solutions are given with respect to the boundary conditions on the characteristic surface (ct − z = 0). They extend previously published results, in particular, localized waves obtained by Hillian [3, 4, 10], to the focus wave modes for lossy media. Higher order modes are obtained in an explicit form.

(iii) We discuss two methods for constructing localized electromagnetic waves from scalar FWM. The first one is based on the vector Hertz-potential technique. Although this method is discussed in many papers on FWM, see, for example, ref. 2, including those devoted to the lossy-medium case [6, 7] the additional treatment is pertinent owing to the fact that in the case of conducting media the general algorithm should be modified [12]. The second method is based on transverse electric (TE) and transverse magnetic (TM) field representations via two scalar functions, which are analogues of Whittaker’s potentials [13, 14] (similar investigations were carried out by Butanen [15] for the axisymmetric TM wave). For TM waves, this method makes it possible to consider some cases when the density of the induced charge is not zero.

2. Scalar waves for conducting media

Let us start with the equation describing the behavior of a scalar-wave function \( \Psi \) in the lossy medium

\[
\left( \nabla^2 - \frac{\partial^2}{\partial \tau^2} - \frac{4\pi}{c} \frac{\partial}{\partial \tau} \right) \Psi = 0
\]

(1)

where \( \tau = ct \) is the time variable. When this equation is applied to electromagnetic waves, \( c \) corresponds to the velocity of light; \( \sigma \) is the conductivity of the medium (we assume Gaussian units, \( \epsilon = \mu = 1 \)). We can rewrite (1) in the cylindrical coordinates \( \rho, \varphi, z \) as

\[
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} - \frac{4\pi \sigma}{c} \frac{\partial}{\partial \tau} \right] \Psi = 0
\]

(2)

We shall investigate the subset of the solutions of (2), which can be represented in the form

\[
\Psi(\rho, \varphi, z, \tau) = e^{ik(\tau + z)} \Phi(\rho, \varphi, \tau - z)
\]

(3)

where \( i = \sqrt{-1} \), \( k \) is a positive constant parameter, and \( \Phi \) is an arbitrary function. Then (2) together with (3) yield

\[
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} - 4(ik + \Omega) \frac{\partial}{\partial \xi} - 4ik\Omega \right] \Phi = 0
\]

(4)

where \( \Omega = \pi \sigma/c \) and \( \xi = \tau - z \). At \( \xi = 0 \) let \( \Phi \) have the transverse variation

\[
\Phi(\rho, \varphi, \xi = 0) = f(\rho, \varphi)
\]

(5)

\( f(\rho, \varphi) \) is some given function. Representing \( \Phi \) by the series

\[
\Phi(\rho, \varphi, \xi) = \sum_{m = -\infty}^{+\infty} \Phi_m(\rho, \xi) e^{im\varphi}
\]

(6)

and, then using the Fourier–Bessel transform

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\(^1\)Received at NRC August 16, 1994.
\[ \Phi_m(s, \xi) = \int_0^\infty \Phi_m(p, \xi) J_m(ps) \, dp \]

(here \( J_m(ps) \) is the Bessel function of the first kind of order \( m \)), one can reduce (4) and (5) to the ordinary differential equation

\[ \left[ s^2 + 4(ik + \Omega) \frac{d}{d\xi} + 4ik\Omega \right] \Phi_m = 0 \]  

(7)

with the condition

\[ \Phi_m(s, \xi = 0) = \tilde{\Phi}_m(s) \]

\[ \Phi_m(s, \xi) \equiv \int_0^s f_m(p') J_m(p's) \, dp' \]  

(8)

where

\[ f_m(p') = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p', \varphi) \, e^{-im\phi} \, d\phi \]

From here on we will assume \( m \) to be non-negative, bearing in mind that the results for the negative values may be obtained by complex conjugation. Applying the Fourier-Bessel transform to the solution

\[ \Phi_m(s, \xi) = \exp\left( \frac{-s^2 + 4ik\Omega}{4(\Omega + ik)} \xi \right) \tilde{\Phi}_m(s) \]  

(9)

of the above problem, we get

\[ \Phi_m(p, \xi) = \exp\left( -\frac{ik\Omega}{\Omega + ik} \xi \right) \]

\[ \times \int_0^s f_m(p') W(p, p') \, dp' \]  

(10)

where

\[ W(p, p') = \int_0^p \exp\left( -\frac{1}{4} \frac{\xi^2}{\Omega + ik} \right) \]

\[ \times J_m(p's) J_m(p's) s \, ds \]

For \( \xi > 0 \) one can simplify \( W(p, p') \). (See item 13.31 in ref. 16) and obtain

\[ \Phi_m(p, \xi) = \frac{2(-i)^m(\Omega + ik)}{\xi} \exp\left( -\frac{ik\Omega}{\Omega + ik} \xi + (\Omega + ik) \frac{p^2}{\xi} \right) \]

\[ \times \int_0^p f_m(p') \exp\left( -(\Omega + ik) \frac{(p')^2}{\xi} \right) J_m\left( \frac{2i(\Omega + ik)}{\xi} pp' \right) \, dp' \]  

(11)

Thus, (11) together with (6) yield the algorithm that enables us to calculate the wave function of type (3), which satisfies the arbitrary specified initial condition \( \Phi(p, \varphi, \xi = 0) = f(p, \varphi) \).

### 3. Focus wave modes

The above method is flexible enough to fit the general scheme for constructing FWM suggested by Sezginer [2]. To obtain the full analogues of Brittingham’s FWM, let us treat \( f(p, \varphi) \) as the waist profile and choose it to be a Fourier series with the coefficients

\[ f_m(p) = A_m \left( \frac{p}{b} \right)^m e^{-\varphi/\alpha} \]  

(12)

where \( A_m, \alpha, \) and \( b \) are an arbitrary complex and two positive real constants, respectively. Then, using the result 6.6.31 from ref. 17, we can calculate the integral in (11) to obtain

\[ \Phi_m(p, \xi) = A_m \left( \frac{p}{b} \right)^m \left( \frac{a^2(\Omega + ik)}{a^2(\Omega + ik) + \frac{\xi^2}{\xi}} \right)^{m+1} \exp\left( -\frac{ik\Omega}{\Omega + ik} \xi - \frac{\Omega + ik}{a^2(\Omega + ik) + \xi^2} \right) \]  

(13)

Finally, for the \( m \)th mode of the wave function one has

\[ \Psi_m(p, \varphi, z, \tau) = e^{im\varphi + ik(\tau + z)} \Phi_m(p, \tau - z) \]

\[ = A_m \left( \frac{p}{b} \right)^m \left( \frac{a^2(\pi\sigma/c + ik)}{a^2(\pi\sigma/c + ik) + (\tau - z)} \right)^{m+1} \exp\left( im\varphi + ik(\tau + z) - \frac{ik\pi\sigma/c}{\pi\sigma/c + ik} (\tau - z) \right) \]

\[ - \frac{\pi\sigma/c + ik}{a^2(\pi\sigma/c + ik) + (\tau - z)^2} \rho^2 \]  

(14)

and

\[ |\Psi_m| = |A_m| \left( \frac{p}{b} \right)^n \left( \frac{\sigma_0^2 + 1}{(\sigma_0 + \xi_0)^2 + 1} \right)^{m+1/2} \exp\left( -\frac{\sigma_0^2 (ka)^2}{\sigma_0^2 + 1} \xi_0 + \frac{\sigma_0^2 + \sigma_0 \xi_0 + 1}{(\sigma_0 + \xi_0)^2 + 1} a^2 \right) \]  

(15)
Re $\Psi_m = |\Psi_m| \cos \left( \arg A_m + m\varphi + (m + 1) \arg(\sigma_0^2 + \sigma_0 \xi_0 + 1 + i\xi_0) - \frac{\sigma_0^2 k a^2}{\sigma_0^2 + 1} \xi_0 - \frac{1}{(\sigma_0 + \xi_0)^2 + 1} \xi_0^2 \frac{\rho^2}{a^2} \right) \right)

where $\sigma_0 = \pi \sigma/(k_c)$ and $\xi_0 = (\tau - z)/(k a^2)$.

For vanishing conductivity and $b = a$, this representation of FWM coincides with Sezginer’s result (eq. (32) in ref. 2). The dependences $\Psi_m(kp, \xi_0)$ for different values of the dimensionless conductivity parameter $\sigma_0$ are plotted in Fig. 1. For the small values of conductivity, the degree of focusing increases with $\sigma_0$ and attains its maximum in the vicinity of $\sigma_0 = 1$ owing to the damping factor $\exp[-(\sigma_0/(\sigma_0^2 + 1)) k(\tau - z)]$. As $\sigma_0$ increases from 1 to infinity, the significance of the above factor is diminished as well as the influence of another damping factor standing in front of the exponent in (15). Due to the latter factor, $\Psi_m$, for $\sigma >> 1$, decays along the $\tau - z$ direction even more slowly than that for $\sigma_0 = 0$. This behavior may seem to be unnatural. However, it results from the fact that we fix the condition on the hyperplane $\tau - z = 0$ and construct only the steady-state solution of definite type. The plots of $\text{Re } \Psi_m$ for two FWM of higher orders are shown in Fig. 2. Although their value for $\rho = 0$ is zero, such solutions may give significant splash (significant value of a wave within a small area) for $m \geq 3$ owing to the factor standing in front of the exponent in the initial condition (12).

Some common differences between FWM for lossy and nonlossy media, damping factor, and specific modulation, are discussed in ref. 8. The focus wave solutions of higher orders are subjected to additional plane-wave modulation of the type $\exp[i(m + 1)\Gamma]$, where the angle variable $\Gamma = \arg(\sigma_0^2 + \sigma_0 \xi_0 + 1 + i\xi_0)$ varies from 0 to $\arctan \sigma_0^2$ as $\xi_0$ increases from 0 to infinity.

When the Gaussian term in (12) tends to the uniform distribution, i.e., when $a \to \infty$, (14) leads to
\[ \Psi_m(p, \varphi, z, \tau) = A_m \left( \frac{p}{b} \right)^m \exp \left[ -\frac{\pi \sigma}{kc} k(\tau - z) \right] \exp \left[ \frac{im \varphi + i\kappa}{2} \left( \frac{\pi \sigma}{kc} \right)^2 + 1 \right] \left( \frac{\pi \sigma}{kc} \right)^2 + 1 \left( \tau - z \right) \right] \]

4. Electromagnetic-field representation via the Hertz potentials

All the above results correspond to the scalar waves that are solutions of (1). To construct the electromagnetic counterparts of the waves in question, which are solutions of the homogeneous Maxwell equations for a medium with a constant conductivity, one may treat \( \text{Re} \Psi_m \) as a value of the Hertz potential.

To obtain a TM component of the electromagnetic field \((E, B)\), one can define the electric Hertz potential \(\Pi\) to get [12]

\[
\begin{align*}
E & = \text{grad} (\text{div} \Pi) - \frac{\partial^2 \Pi}{\partial \tau^2} - \frac{4\pi \sigma}{c} \frac{\partial \Pi}{\partial \tau} = \text{curl} \left( \text{curl} \Pi \right) \\
B & = -\text{curl} \left( \frac{4\pi \sigma}{c} \Pi \right)
\end{align*}
\]

(18)

A TE component of the field may be constructed with the help of the magnetic Hertz potential \(\Pi^*\) (see the last part of item 1.11 in ref. 12 for details).

\[
\begin{align*}
E & = -\text{curl} \frac{\partial \Pi^*}{\partial \tau} \\
B & = \text{grad} (\text{div} \Pi^*) - \frac{\partial^2 \Pi^*}{\partial \tau^2} - \frac{4\pi \sigma}{c} \frac{\partial \Pi^*}{\partial \tau} = \text{curl} \left( \text{curl} \Pi^* \right)
\end{align*}
\]

(19)

Little attention is paid in the literature to the fact that the field vectors must also satisfy two scalar Maxwell equations and the law of conservation of charge. The curl-type solutions (18) and (19) (i) are easily seen to fit all these equations and (ii) do not produce the induced charge \(q = \frac{1}{4\pi} \text{div} E = 0\). The last result may be treated as a constraint for the type of possible electromagnetic problems where waves that can be described by a Hertz potential of the above type are expected to appear.

Electromagnetic waves in conducting media can be also described in terms of a one-component Hertz vector by analogy with [13, 18]. A TM wave can be represented via the electric Hertz vector \(\Pi = e_\Pi\)

\[
\begin{align*}
E_x & = \frac{\partial^2 \Pi}{\partial \varphi^2}, \quad E_y = \frac{1}{\rho} \frac{\partial \Pi}{\partial \varphi}, \quad E_z = -\frac{\partial^2 \Pi}{\partial \tau^2} - \frac{4\pi \sigma}{c} \frac{\partial \Pi}{\partial \tau} + \frac{\partial^2 \Pi}{\partial z^2} \\
B_x & = \frac{1}{\rho} \frac{\partial}{\partial \varphi} \left( \frac{\partial \Pi}{\partial \tau} + \frac{4\pi \sigma}{c} \Pi \right), \quad B_y = -\frac{\partial}{\partial \rho} \left( \frac{\partial \Pi}{\partial \tau} + \frac{4\pi \sigma}{c} \Pi \right), \quad B_z = 0
\end{align*}
\]

(20)
while for a TE wave one can obtain

\[ E_\rho = -\frac{1}{\rho} \frac{\partial^2 \Pi^*}{\partial \rho \partial \tau}, \quad E_\varphi = \frac{\partial \Pi^*}{\partial \sigma \varphi}, \quad E_z = 0 \]
\[ B_\rho = \frac{\partial \Pi^*}{\partial \rho \varphi}, \quad B_\varphi = \frac{1}{\rho} \frac{\partial \Pi^*}{\partial \rho \sigma}, \quad B_z = \frac{\partial^2 \Pi^*}{\partial \tau^2} + \frac{4 \pi \sigma}{c} \frac{\partial \Pi^*}{\partial \tau} + \frac{\partial^2 \Pi^*}{\partial z^2} \] (21)

where \( \Pi^* = e_\rho \Pi^* \) is the magnetic Hertz vector.

Substituting (20) and (21) into the vector’s Maxwell equations, one can check that they are satisfied but the one for the \( z \) components leads to a scalar equation of type (1).

\[ \nabla \cdot \begin{pmatrix} v \\ v^* \end{pmatrix} - \frac{\partial^2}{\partial \tau^2} \begin{pmatrix} v \\ v^* \end{pmatrix} - \frac{4 \pi \sigma}{c} \frac{\partial}{\partial \tau} \begin{pmatrix} v \\ v^* \end{pmatrix} = 0 \] (22)

for the functions

\[ v = \frac{\partial \Pi}{\partial \tau} + \frac{4 \pi \sigma}{c} \Pi \quad \text{and} \quad v^* = \frac{\partial \Pi^*}{\partial \tau} \]

For TM waves the above representation (i) enables us to consider some problems where the density of the induced charges is not zero and (ii) can be generalized to the inhomogeneous Maxwell equations with the extrinsic current density parallel to the \( z \) axis \( J_{\text{ex}} = e_\rho J_{\text{ex}} \) (the values of densities of the extrinsic current \( J_{\text{ex}} \) and charge \( q_{\text{ex}} \) must satisfy the law of conservation.

\[ c \frac{\partial q_{\text{ex}}}{\partial \tau} + \text{div} \ J_{\text{ex}} = 0 \]

For TE waves (20) gives the zero density of the induced charge \( \text{div} E / 4 \pi \). One can check by direct calculation that the constraint of conservation of the induced charge

\[ c \frac{\partial}{\partial \tau} \left( \frac{1}{4 \pi} \text{div} E \right) + \text{div} (\sigma E) = 0 \]

holds for both TM and TE waves. It is easily seen that if \( \Pi \) is the solution of the equation

\[ \nabla^2 \Pi - \frac{\partial^2}{\partial \tau^2} \Pi - \frac{4 \pi \sigma}{c} \frac{\partial}{\partial \tau} \Pi = 0 \]

then \( \nu \) is the solution of (22), but in this case \( \nu = 0 \). Hence, if we are able to satisfy the initial and (or) boundary conditions, we can restrict ourselves to the potential \( \Pi \) solving the above simplified equation instead of (22). To obtain Brittingham’s TM FWM of order \( m \), one should choose a one-component Hertz vector in the form \( \Pi = (\text{Re} \Psi_m) e_z \) and use formulas (20). The FWM of TE type may be obtained with the help of \( \Pi = (\text{Re} \Psi_m) n \), where \( n \) is the arbitrarily directed constant vector.

5. Conclusion

In case of the initial condition \( f(r, \varphi) = \text{const} \), which corresponds to \( m = 0 \) and \( a \to \infty \), (17) gives

\[ \Psi_{\psi} (r, \tau) = \lim_{a \to \infty} \Psi_{\psi} (r, \varphi, z, \tau) = A_0 \exp \left[-\frac{k^2 \pi \sigma}{(\pi \sigma/c)^2 + k^2} (\tau - z) \right] \exp \left[i k \left( \tau + z - \frac{(\pi \sigma/c)^2}{(\pi \sigma/c)^2 + k^2} (\tau - z) \right) \right] \] (23)

The function \( \Psi_{\psi} (r, \tau) \) is the solution of the telegraph equation

\[ \frac{\partial^2 \Psi_{\psi}}{\partial z^2} - \frac{\partial^2 \Psi_{\psi}}{\partial \tau^2} - \frac{4 \pi \sigma}{c} \frac{\partial \Psi_{\psi}}{\partial \tau} = 0 \] (24)

with the condition

\[ \Psi_{\psi} \big|_{\tau - z = 0} = A_0 \exp [i k (\tau + z)] \] (25)

Equation (23) is consistent with the description of the interaction between plane electromagnetic waves and the conducting half-space \( \tau - z > 0 \). For example, such a simple model can be taken for the approximate description of the ionization of a medium by an X-ray pulse. The front boundary of the ionization domain moves with the velocity of light, and in this case the \( z \) component of the electric field is the solution of the ordinary wave equation for \( \tau - z < 0 \). In the domain \( \tau - z > 0 \), we have the conducting ionization area, a medium whose electro- magnetic properties can still be satisfactorily described only by inserting the additional term

\[ \frac{4 \pi}{c^2} j = \frac{4 \pi}{c} \alpha E \]

into the corresponding Maxwell equation [12], so for \( \tau - z > 0 \), the \( z \) component of the electric field can be described by the solution of (24). Although the phase velocity differs from \( c \), the propagation speed of wave fronts (which is determined entirely by the coefficients of higher derivatives in the equation for the wave function) remains equal to \( c \), see Chap. V of ref. 12 for details. When the incident wave is sinusoidal with the step-function envelope, both the transient and steady-state resulting waves can be described in terms of Lommel’s functions of two variables [19, 20].

The steady-state wave solution is nearly the same as (23) except for the coefficient and differs from functions of the type

\[ \exp \left[i k \left( \tau \pm z \sqrt{\frac{1 - 4 i \pi \sigma}{k^2 c}} \right) \right] \]

which are traditionally used for the description of the plane waves in conducting media [21]. Thus, the solution obtained is supposed to be produced as a kind of steady-state wave resulting from the interaction between electromagnetic fields and a
conducting half-space whose boundary moves with the velocity of light. Some differences between (40) and the results given in refs. 19 and 20 have their origin in the conditions on the plane \( \xi = \tau - z = 0 \). Here, we have constructed the analogues of Brittingham’s focus wave modes choosing the condition (21), which implies continuity of the wave function at \( \tau - z = 0 \). In refs. 19 and 20, the discontinuity of the field at the boundary is taken into consideration. This discontinuity seems to be important when the real possibilities for the FWM generation are discussed.