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Statistical characteristics of photon paths and optimization of the tomography algorithms for the case of strongly scattering media

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ABSTRACT

The theoretical and experimental studies were carried out for statistical characteristics of photon paths in strongly scattering media dependently on type of inhomogeneities, boundary conditions and method of measurements (time-gating and frequency-domain). The possibility to represent the signal perturbations due to the macroinhomogeneities as an integral along the mean photon path is used to solve the tomography reconstruction problem in terms of volume quantization. The optimum quantization scale is chosen on the basis of area across which the macroinhomogeneity characteristics are averaged.

Key words: diffusion approximation, photon mean path, tomography reconstruction algorithm

1. INTRODUCTION

It is well known that any tomographical study includes two problems to be solved in tandem. The first one, forward problem, implies that measured signals and their variations must be represented as integrals along known trajectories between signals source and receiver, and the second one, inverse problem, is the reconstruction of the internal inhomogeneities locations and characteristics proper using the forward problem solutions for a number of different signal trajectories.

In case of signal strong scattering the main difficulty for tomographical studies lies in obtaining a reliable information about source-detector (SD) signal trajectories and their statistical characteristics, such as spatial distribution of probability density for different SD paths and its time behavior.

On the base of carried out theoretical and experimental studies here we expound the solution algorithms for strongly scattering objects optical tomography applicable to different methods of time-resolved measurements (time-gating and frequency domain), different boundary geometries of objects and different types of macroinhomogeneities under investigations (such as absorptive, dispersive, refractive and of combined types).
2. TERMS AND METHODS

2.1 Main equation

Diffusion equation for photon density \( \phi(r,t) \) in the strongly scattering media is

\[
\frac{n(r)}{cD(r)} \frac{\partial}{\partial t} \phi(r,t) - \Delta \phi(r,t) + \frac{\mu_a(r)}{D(r)} \phi(r,t) = \frac{S(r,t)}{D(r)}. \tag{1}
\]

where \( c \) is the light velocity, \( n \) is the refractive index, \( D = \{3(\mu_a + (1-g) \mu_s)\}^{-1} \) is the diffusion coefficient, \( \mu_a \) is the absorption coefficient, \( \mu_s \) is the scattering coefficient, \( g \) is the average cosine of the scattering angle, \( S(r,t) \) is the function of source distribution, \( r \) is the radius vector.

2.2 Time representation

For instantaneous point source the solution of equation (1) can be represented as the sum

\[
\phi_0(r,t) + \phi_1(r,t), \tag{2}
\]

where \( \phi_0(r,t) \) is the eq.(1) solution for homogeneous bulk medium with constant parameters \( (D_0, n_0, \mu_{0a}) \) and \( \phi_1(r,t) \) is the change in photon density induced by inhomogeneity and fits the equation

\[
\frac{n_1}{cD_0} \frac{\partial}{\partial t} \phi_1(r,t) - \Delta \phi_1(r,t) + \frac{\mu_{0a}}{D_0} \phi_1(r,t) = \frac{S_1(r,t)}{D_0}, \tag{3}
\]

where \( S_1(r,t) = \frac{(\mu_{0a} - \delta \mu_a)}{D_0} \frac{\partial}{\partial t} \phi_0(r,t) + \frac{n_0}{cD_0} \frac{\partial}{\partial t} \ln \phi_0(r,t), \delta n = n(r) - n_0, \delta \mu_a = \mu_a(r) - \mu_{0a}. \)

The necessary condition for such representation is \( |\phi_0(r,t)| \phi_1(r,t) \), that is, the inhomogeneities are bound to be of weak contrast. (Other case is not of interest because of inhomogeneities would be visible through the object or would give significant shadows at the detector when sounding.)

The solutions \( \phi_0(r,t) \) and \( \phi_1(r,t) \) have the remarkable features:

\[
\phi_0(r,t) = \int S_1(r_1,t)G(r-r_1,t-\tau) d^3\xi, \tag{4}
\]

\[
\frac{\phi_1(r,t)}{\phi_0(r,t)} = \int_0^\infty \int S_1(r_1,t) \frac{\phi_0(r_1,t)}{\phi_0(r,t)} G(r-r_1,t-\tau) d^3\xi, \tag{5}
\]

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and
\[
\nabla \cdot \varphi(t, r) = \frac{\rho}{\bar{n}_0} \int_\tau^t \nabla \cdot \int S(t, r) \nabla \cdot (G(r-t, r) \nabla G(r-t, r)) \, dt \, d^3 \tau,
\]
(6)

where \( G \) is the Green function, \( r_i \) and \( r \) are the integration variables.

One can see from (5) and (6) that unity normalized functions
\[
\begin{align*}
\frac{\varphi(t, r) G(r-t, r)}{\int \varphi(t, r) G(r-t, r) \, d^3 \tau} \quad \text{and} \quad \frac{\varphi(t, r) \nabla G(r-t, r)}{\int \varphi(t, r) \nabla G(r-t, r) \, d^3 \tau}
\end{align*}
\]

have a clear meaning as probability density distribution of photons passing through the inhomogeneity to be at the intermediate instant \( r \) at the same point \( r_i \), where they would be in the absence of the inhomogeneity.

Thus, the changes both in photon density value and in signal intensity at the detector are defined by the statistical averaging of the inhomogeneities parameters over the spatial area of photon paths from the source to the detector.

It is notable that there exists the possibility to separate out the positions of maxima of mention above unity normalized functions in relation to intermediate time \( \tau \) \((0 \leq \tau \leq t)\). So, integration over time in (5) and (6) is turned out to be integration over the path of photon density distribution (PDD) maximum. At the same time the parameters of the inhomogeneity inside the area of its intersection with the volume of PDD are averaged along this path correspondingly to the current width of the PDD at current instant \( \tau \).

The integrals in (5) and (6) acquire a form
\[
\int_\tau^t \left( \frac{\mathcal{S}^i_\tau(r)}{\mathcal{R}(r)} \right) \, dt,
\]
where \( \mathcal{R}(r) \) is the trajectory function of PDD maximum. \( \mathcal{S}^i_\tau(r) \) is the inhomogeneity function averaged along the trajectory.

### 2.3. Frequency representation (photon density waves)

In the case of sine-shaped intensity-modulated point source of light the function \( S(t, r) \) in equation (1) is
\[
S(t, r) = (S_0 + S_1 \cos \omega t) \mathcal{R}(r),
\]
(7)

where \( \omega \) is the angular modulation frequency of the source, and the solution can be represented as
\[
\varphi(t, r) = \varphi_s(t, r) + \varphi_p(t, r) e^{-i \omega t},
\]
(8)

So, the frequency-dependent equation for \( \varphi_p(t, r) \) will be as follows
\[
-\frac{i \omega}{c D(r)} \varphi_p(t, r) - \Delta \varphi_p(t, r) + \frac{\mu_p(t, r)}{D(r)} \varphi_p(t, r) = \frac{\delta(0)}{D(r)}.
\]
(9)

Writing as above \( \varphi_p(t, r) = \varphi_{0p}(t, r) + \varphi_{1p}(t, r) \) we obtain the equation for \( \varphi_{1p}(t, r) \):

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where the inhomogeneity function is
\[ F(r, \omega) = \left( \frac{\mu_0}{D_0^2} \frac{\partial n_0}{\partial D_0} - i \omega n_0 \frac{\partial}{\partial D_0} \right) = 2k_0 \Re k_0^2 = \frac{i \omega n_0}{D_0} \frac{\mu_0}{D_0} \, . \]

Within Huygens-Fresnel approximation the frequency analogs of (4), (5) and (6) are

\[ \varphi_0 (r, \omega) = \frac{k_0}{2\pi} \int G_n (r - r_1) d^2 r_1, \quad \text{where} \quad G_n (r) = \frac{1}{|r|} \exp (ik_0 |r|) \, , \quad (11) \]

\[ \varphi_1 (r, \omega) = \frac{i}{\varphi_0 (r, \omega)} \int \frac{\varphi_0 (r_1, \omega) G_n (r - r_1)}{\varphi_0 (r_1, \omega) G_n (r - r_1)} d^2 r_1, \quad (12) \]

\[ \frac{\nabla \varphi_1 (r, \omega)}{\nabla \varphi_0 (r, \omega)} = \frac{\varphi_0 (r, \omega) \nabla G_n (r - r_1)}{\varphi_0 (r, \omega) \nabla G_n (r - r_1)} d^2 r_1. \quad (13) \]

In (12) and (13) functions \( \frac{\varphi_0 (r_1, \omega) G_n (r - r_1)}{\varphi_0 (r_1, \omega) G_n (r - r_1)} \) and \( \frac{\varphi_0 (r, \omega) \nabla G_n (r - r_1)}{\varphi_0 (r, \omega) \nabla G_n (r - r_1)} \) has the probabilistic meaning similar to that of the above discussion in 2.2. So, the integrals in (12) and (13) also can be transformed to the trajecotoric form.

2.4. On the similarity of time and frequency representations

In the case of infinite scattering medium we have in time terms

\[ \langle \mathcal{S}_1 (R(\tau)) \rangle = \int \mathcal{S}_1 (R(\tau) + \Delta r_1) \left[ 4\pi D_0 \frac{C}{n_0} \exp \left( - \frac{\Delta r_1^2}{4 D_0 \Delta t} \right) \right] d^2 \Delta r_1 \quad \text{exp} \left( \frac{\Delta r_1^2}{4 D_0 \Delta t} \right) \] \( \frac{\varphi_0 (r_1, \omega) G_n (r - r_1)}{\varphi_0 (r_1, \omega) G_n (r - r_1)} \) d\( ^2 r_1 \, . \quad (14) \]

where \( R(\tau) = r \neq 0 \), \( \Delta r_1 = r_1 - R(\tau) \), and \( c \Delta t n_0 >> 1 \).

and in frequency terms

\[ \frac{\varphi_0 (r, \omega)}{\varphi_0 (r, \omega)} = \frac{1}{2\pi} \int |r| \, \exp (ik_0 |r|) d^2 r_1. \quad (15) \]

where \( |k_0 r| >> 1 \).

It is seen from (14) and (15), that PDD maximum trajectory or, in other words, photon mean path (PMP) is the straight line between source and detector.

The PDD maximum widths at the level \( \exp (-1) \), \( \rho_0 \), defined by the value of photon mean square deviation from the mean path and corresponded to (14) and (15) are
\[ p_r(R(\tau)) = 4 \sqrt{\frac{D_0}{n_0}} \frac{(|r| R(\tau)|R(\tau)|)}{|r|^2}, \]  
\[ p_r(R_w) = 4 \sqrt{\frac{D_0}{n_0}} \frac{T (|r| R_w)}{|r|^2}, \]  
(16)  
(17)

Here \( R = \frac{c' r}{|r|} \) and \( R(\tau) \) are responding to current location of PDD maximum on its trajectory. \( T = \frac{|r|}{v} \) and \( v = \sqrt{2\omega D_0 n_0} \) is the velocity of photon density wave.

The uniform appearance of (16) and (17) suggests that for both types of representation the analysis of PMP behavior and dimensions of photon spreading zone can be described with equal facility using time-terms only substituting in frequency-domain case \( \frac{|F|}{\nu} \) for \( t \), \( \tau \) for \( \frac{|F|}{\nu} \) and \( \frac{|F_w|}{\nu} \) for \( \tau \).

Note: On the basis of our previous study carried out by MC numerical simulations, we imply that \( t \) is not the time interval between light "delta"-pulse and instant of observation, but it is the value of time delay between photon direct flight duration \( \frac{|r|}{\nu} \) and instant of observation. This must be taken into account especially when using small \( |r| \) and \( t (|r| < 30D_0; c|t| < 2|r|/n_0) \).

3. RESULTS AND ANALYSIS

Forward problem

3.1. Trajectoryic history

By the above described general approach we have analyzed the photon mean paths PMP and their mean square deviation from it \( p_r(\tau) \) for different boundary conditions of strongly scattering medium. The analysis showed some regular trends which are as follows:

i) For all cases at the initial (source) and terminal (detector) points of PMP \( p_r(\tau=0) \) and \( p_r(\tau=t) \) are equal to 0.

ii) The PMP lies in the same plane with the straight line connecting the points of source and detector

iii) The maximum of photon density distribution moves along the PMP in general, unevenly, but its projection on the source-detector straight line moves (at first approximation at least) at steady rate \( \frac{|r|/t}{\tau} \) which is more less than \( c/n_0 \).

3.2 Detectability and sensitivity to macroninhomogeneities

Let the inhomogeneity is located in point \( r \), and its dimension is \( \Delta r \). The condition for optimum detectability are evidently as follows
i) $R(\tau) = r_i$ and $p(x) = \Delta r_i$, were $r_i$ is instant of "meeting" the inhomogeneity with PDD maximum moving along mean trajectory. These conditions can be realized by optimum choice of the positions of source and detector and of the value of $t$.

ii) The minimum detectable values of $S_i$ for every real measuring system depends on minimum of admissible signal-to-noise ratio. It can be calculated by using formulas (6) and (15) with taken into consideration the conditions from i).

Note: For the calculation simplicity it is useful to represent $S_i$ in Gauss form:

$$S_i(\tau) = S_0(\tau) \exp(-2|\tau - r_i|/\Delta r_i)^2$$

3.3. Trajectory forms of PMP in different boundary conditions.

For the purpose of formulas simplification the consideration below is pursued in the plane ZX with source coordinates $(0,0)$ and detector coordinates $(0,x_d)$.

3.3.1. Infinite medium

As it follows from (14) and (15) in this case $R(\eta) = X(\eta) = x_d\eta$, where $X$ is current trajectory coordinate, and the PDD maximum trajectory formula is

$$x_d(X) = X$$

So, the trajectory is the straight line between source and detector (Figure 1), the PDD maximum's velocity is equal to $x_d\eta$ and its width varies on $X$ as $p$, in (16) (dash curve lines on the figure)

$$p_d(X) = \frac{4D_0}{\sqrt{n_s}} t \frac{(X - x_d)X}{x_d^2}$$ (18)

3.3.2. Sounding of semi-infinite medium

The medium is situated at $z \geq 0$ and bounded at $x_d(X) = 0$.

$$x_d^2(X) = \frac{P_d(X)^2}{2} = 4D_0 \frac{x_d^2}{n_s} \frac{(X - x_d)X}{x_d^2}$$

The trajectory is ellipse with semi-axes $x_d/2$ and

$$\sqrt{D_0 \frac{x_d}{n_s} t}$$ (Figure 2).
3.3.3 Cylindrical form of bound

\[ z_s(X) = \sqrt{R_e^2 - \frac{x_d^2}{4}} - \sqrt{R_e^2 - (X - \frac{x_d}{2})^2}, \text{ where } R_e \text{ is the radius of the bound} \]

\[ z_s'(X + \Delta X) \approx (\Delta z - \Delta R)^2 + \frac{R_e^2 - (X + \frac{x_d}{2})^2}{R_e^2 - X(x_d - X)} \]

where \( \Delta R = R_e - \sqrt{R_e^2 - X(x_d - X)} \),

\[ \Delta z = \frac{1}{2} \left[ 2R_e - \sqrt{\Delta R^2 + \left( \frac{P_r}{2} \right)^2 - \gamma^2 (2R_e - \Delta R)^2 + \left( \frac{P_r}{2} \right)^2} \right]. \]

\[ \Delta X^2 = (\Delta z - \Delta R)^2 + \frac{(x_d/2 - X)^2}{R_e^2 - X(x_d - X)} \text{, and } \frac{P_s(X)}{2} = 4D_0 \frac{\beta}{\theta_0} \frac{(X - x_d)X}{x_d^2} \]

The result was obtained by semi-heuristic way on the base of V.V. Lyubimov's solution for source embedded inside the semi-infinite medium and experimentally tested by 600 MHz - frequency - domain method.

Note: On above figures 1,2 and 3 the parameters for calculations were as follows \( x_d = 150D_0 = 0.144c/\theta_0 \).

The following two bound geometries were investigated experimentally:

3.3.4. Layer with parallel bounds (figure 4)

3.3.5. Rectangular corner (figure 5)

Our analysis has shown that for the cases of complicated bound geometry 3.3.3, 3.3.4 and 3.3.5. It is possible to represent the trajectory curve line as three steps polygonal line, where the first and third steps are normal to bound straight line segments and the third step is rectilinear segment connecting the terminal.
functions $a(x)$ and $b(x)$ and value of $x_p$. Such polygonal approximation (dash lines on the figures) is very convenient and does not decrease the spatial resolution.

**Inverse problem**

Using the cylindrical geometry as an example, solving procedure can be subdivided into the following steps:

1) We direct the pulsed or amplitude modulated radiation beam to the center of the sample and measure the time of beam propagation through the diameter of the sample. This enables us to define the velocity of pulse or diffuse wave propagation in the scattering medium, and then the average diffusion coefficient $D_0$ and parameter $k_0 = \sqrt{(\varepsilon\mu_0}/cD_0 - \mu_0)/D_0$, by the solutions of (1) or (9) for $\varphi_0$.

2) Knowing the refraction index $n_0$ and the diffusion coefficient $D_0$ of undisturbed medium, one can calculate, with (18) and (19) the averaged photon trajectory and mean square deviation of photon paths from this trajectory for given locations of the radiation source and detector. This enables us to estimate the resolution parameter $p_0$, defining our choose the dimensions of primary elements (cells) to the solving scheme of the tomography reconstruction problem. According to the scheme, the total volume of the cylindrical area to be scanned should be broken down into cylindrical layers whose thickness must correspond to the dimension of the cell $r_c$.

3) We begin with probing of the outer layer, spacing the source and the detector so that the photon trajectory lies entirely within one cell (Figure 6). By this means for each cell in the layer one can define experimentally the parameter $k_{b_0}$ (or $D_{a_1}$), where $i = 1, 2, \ldots, m_1$ is the serial number of the cell. If the radius of the scanned cylindrical area is $R_c$, then the length of the first (outer) layer is $2\pi R_c$ and the total number of cells in the layer is $2\pi n$, where $n = R_c/r_c$ is the total number of layers. Shifting both the source and detector by the same length $r_c$, and making $m_1$ measurements, we define parameters $(k_{b_0})_{i=1}^{m_1}$ for each cell of the first layer.

![Figure 6](image1)

4) When coming to the next layer, it is necessary to choose, with the help of, (19) the distance between the source and detector in such a way as the upper point of average trajectory to lie in the middle of the second layer (Figure 7). Here the experimental data are functions of already defined parameters $k_{a_1}$ of several cells on the first layer and of parameters to be defined $k_{a_1}$ of several cells on the second layer. As far as the radius of the second layer is $R_c + r_c$, its length is $2\pi (R_c + R_c)$, and the number of cells on this layer is $m_2 = 2\pi (R_c + R_c)/r_c = 2\pi (n+1)$. Carrying out measurements for each cell of the second layer, one obtains $m_2$ simultaneous equations with $m_1$ unknown parameters $(k_{a_1})_{i=1}^{m_2}$. 5) Methods of stage 4 should be successively applied to the next layers until one gets the total set of parameters $k_{a_1}$:

$$(k_{a_{11}}, k_{a_{12}}, \ldots, k_{a_{1m}})$$
5. CONCLUSION

We can conclude that above described general approach pioneered by V.V.Lyubimov for trajectory solutions of the diffusion equation for photon density in a strongly scattering medium\(^2,3,4\) allows to obtain the reliable a priori information about the forms of photon mean paths in object of different geometries. This, in turn, leads to very simple algorithms for optical tomography reconstruction of objects internal structure. The approach is applicable to the next, more exact trajectory approximation when analyzing of the trajectories of "center of weight" of PDD. In this analysis it is necessary to take into consideration the fact that such a center participates simultaneously in two independent movements: the first is uniform movement along the source-detector straight line and the second is repelling with variable speed from medium bounds to the point of balance.

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7. REFERENCES


