Waves produced by a travelling line current pulse with high-frequency filling

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A model of the travelling-wave line source of electromagnetic waves is extended to the case of a line current pulse with high-frequency filling. No limitation on the modulation phase velocity is assumed and peculiarities of the emanated waves corresponding to subluminal, luminal, and superluminal phase velocities of modulation of the source current are discussed.

INTRODUCTION

The objective of the present study is to extend a model of the travelling-wave line source of the electromagnetic waves [1, 2] to the case of a line current pulse with high-frequency filling. The source current pulse starts its motion at some fixed moment of time, from here on taken as the origin of the time coordinate, \( \tau = 0 \) \((\tau = ct , c \) is the velocity of light), along the line-segment radiator of length \( l \), which governs the choice of the cylindrical coordinate system \( \rho, \varphi, z \) in such a way that its origin, coinciding with the current-pulse starting point, has the coordinates \( \rho = 0, z = 0 \) and its end has the coordinates \( \rho = 0, z = l \). As so, the current density vector can be described as

\[
j = e_z j_z, \quad j_z = \begin{cases} \frac{\delta(\rho)h(z)}{2\pi \rho} & \tau > 0 \\ 0 & \tau < 0 \end{cases}, \tag{1}\]

where \( \delta(\rho) \) is the Dirac delta function; the step function \( h(z) = \begin{cases} 1 & z > 0 \\ 0 & z < 0 \end{cases} \) and the analogous step function \( h(l-z) \) are used for explicit representation of the radiator finiteness while \( I(z, \tau) \) denotes the line-current distribution, which in the case of a modulated pulse whose front and back move with a subluminal or luminal velocity \( v_{\text{front}} = \beta c \) \((0 < \beta \leq 1) \) takes the form

\[
I_{\pm} = h(\beta \tau - z) h(z - \beta \tau + \beta T) U(z, \tau) M_{\pm}(z, \tau). \tag{2}\]

Here the parameter \( T \) represents the current-pulse duration, a differentiable function \( U(z, \tau) \) the current-pulse envelope, and

\[
M_{\pm} = \exp(i k (v_{\text{phase}} t \pm z)) = \exp(i k (\Omega \tau \pm z)) \tag{3}\]

the factor corresponding to the co-sinusoidal (the real part) and sinusoidal (the imaginary part) modulation with the spatial period \( 2\pi/k \) and the phase velocity \( v_{\text{phase}} = \Omega c \). Throughout this work, subluminal, luminal, and superluminal phase velocities will be discussed, so \( \Omega \) is supposed to vary from 0 to infinity.
Imposing zero initial conditions to the electric induction $\mathbf{D}$ and the magnetic field strength $\mathbf{H}$ and representing these vectors through the Whittaker-Bromwich potential $u$ [3, 4]

\[
D_{\rho} = \frac{\partial^2}{\partial \rho \partial z} u, \quad D_z = -\frac{\partial^2}{\partial \tau^2} u + \frac{\partial^2}{\partial z^2} u, \quad H = H_{\varphi} = -c \frac{\partial^2}{\partial \rho \partial \tau} u,
\]

we obtain from the Maxwell equations the scalar initial-value problem [5]

\[
\left( \frac{\partial^2}{\partial \tau^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) - \frac{\partial^2}{\partial z^2} \right) \Psi = \frac{1}{c} j_z, \quad \Psi \equiv 0 \quad \tau < 0
\]

with respect to the function $\Psi = \partial u/\partial \tau$. Construction of the general solution to the above problem with the help of the Smirnov method of incomplete separation of variables [6], the Fourier-Bessel transform, and the Riemann formula [7] is described in details elsewhere [8]. The final formula is a one-dimensional integral, which in the case of current density (2), (3) takes the form

\[
\Psi_{\pm} = \frac{1}{4\pi c} \int_{z_1}^{z_2} dz' \frac{U \left( z', \tau - \sqrt{\rho^2 + (z - z')^2} \right)}{\sqrt{\rho^2 + (z - z')^2}} \exp \left( ik \left( \tau - \sqrt{\rho^2 + (z - z')^2} \pm z' \right) \right)
\]

where $\Psi_+$ and $\Psi_-$ are the wavefunctions corresponding to the source-current distributions $I_+$ and $I_-$ respectively. The limits of integration $z_{1,2}$ are determined by the current-pulse and radiator parameters $l$, $T$, $\beta$ as well as by the observation time $\tau$ and location $\rho$, $z$. The following different cases, whose physical interpretation is given in [5, 8], can be distinguished:

**case (i)** $0 < \tau - r < T$ and $\tau - r_1 < l/\beta$

if $T + r < l/\beta + r_1$ (short current pulse) then

**case (ii)** $\tau - r > T$ and $\tau - r_1 < l/\beta$

otherwise (long current pulse)

**case (iii)** $0 < \tau - r < T$ and $\tau - r_1 > l/\beta$

**case (iv)** $r < \tau - T < l/\beta + r_1$ and $\tau - r_1 > l/\beta$

Here $r = \sqrt{\rho^2 + z^2}$ and $r_1 = \sqrt{\rho^2 + (z - l)^2}$. The wavefunction is zero for earlier, $\tau - r < 0$, and later, $\tau - T > l/\beta + r_1$, moments of time. Cases (ii) and (iii) are alternative: either the case sequence (i), (ii) and (iv) or (i), (iii) and (iv) is realized for each observation location, depending on whether $T + r$ is less than $l/\beta + r_1$ (short pulse) or not (long pulse). The four cases correspond to different interrelations between the distance from the radiator’s beginning to the observation point $r$, the distance from the radiator’s end to the observation point $r_1$, and the positions of four spherical wavefronts emanated from the radiator beginning and end by the pulse front and back.
Let us represent the current-pulse envelope \( U(z, \tau) \) in the form \( U(z, \tau) = \tilde{U}(\tau - z, \tau + z) \), which for the pulse modulation \( M_\tau \) results in

\[
\Psi_- = \frac{1}{4\pi c} \int_{z_1}^{z_2} U(z, \tau) \frac{\tau - z - \Phi(z)}{(\tau - z - \Phi(z))^2 - \varepsilon \rho^2} \exp(iK[\Phi(z) + \varepsilon s(z)]) \frac{\partial}{\partial \tau} \phi(z) d\tau.
\]

(8)

Changing the integration variable to \( \xi = \tau - \sqrt{\rho^2 + (z - z')^2} - \sqrt{\rho^2 + (z - z')^2} \), one gets

\[
\Psi_- = -i \frac{1}{4\pi c} \int_{\Phi_1}^{\Phi_2} d\xi \tilde{U}(\xi, s) \frac{\tau - z - \xi}{(\tau - z - \xi)^2 - \varepsilon \rho^2} \exp(iK[\Phi(z) + \varepsilon s(z)]) \frac{\partial}{\partial \xi} \phi(z) d\xi.
\]

(9)

where \( \Phi_1 = \Phi_1(z_2) \) and \( \Phi_2 = \Phi_2(z_1) \) are the integration limits \( z_1, z_2 \) recalculated for the new variable \( \xi \),

\[
s(z) = \tau - \sqrt{\rho^2 + (z - z')^2} + z' = \tau + z - \frac{\rho^2}{\tau - z - \xi}, \quad \varepsilon = \frac{1}{2(\Omega + 1)} K = \frac{\Omega + 1}{2}, \quad (11)
\]

Assuming that \( \tilde{U}(\xi, s(z)) \) is a slowly varying function of \( \xi \), \( q(z) = \tilde{U}(\xi, s(z)) \frac{\tau - z - \xi}{(\tau - z - \xi)^2 - \varepsilon \rho^2} \) is the continuous function and \( q'(\xi) \) is the absolutely integrable function, one gets by integration by parts [9] the following estimation

\[
\Psi_- = \frac{1}{4\pi c} \int_{\Phi_1}^{\Phi_2} \tilde{U}(\Phi_1, s(\Phi_1)) \frac{\tau - z - \Phi_1}{(\tau - z - \Phi_1)^2 - \varepsilon \rho^2} \exp(iK[\Phi_1 + \varepsilon s(\Phi_1)]) - \tilde{U}(\Phi_2, s(\Phi_2)) \frac{\tau - z - \Phi_2}{(\tau - z - \Phi_2)^2 - \varepsilon \rho^2} \exp(iK[\Phi_2 + \varepsilon s(\Phi_2)]) + o\left(\frac{1}{K^2}\right).
\]

(11)

Neglecting terms of order \( (K')^{-1} \) and higher, we readily get from (11) the following approximation for the magnetic field strength

\[
H_- = -c \frac{\partial^2 u}{\partial \rho \partial \tau} = -c \frac{\partial^2 u}{\partial \rho^2} \approx \frac{1}{4\pi} \left\{ \tilde{U}(\Phi_2, s(\Phi_2)) \frac{\tau - z - \Phi_2}{(\tau - z - \Phi_2)^2 - \varepsilon \rho^2} \left[ \frac{\partial \Phi_2}{\partial \rho} + \varepsilon \frac{\partial}{\partial \rho} (s(\Phi_2)) \right] \exp(iK[\Phi_2 + \varepsilon s(\Phi_2)]) \right\}
\]

\[+ \tilde{U}(\Phi_1, s(\Phi_1)) \frac{\tau - z - \Phi_1}{(\tau - z - \Phi_1)^2 - \varepsilon \rho^2} \left[ \frac{\partial \Phi_1}{\partial \rho} + \varepsilon \frac{\partial}{\partial \rho} (s(\Phi_1)) \right] \exp(iK[\Phi_1 + \varepsilon s(\Phi_1)]) \exp(iK[\Phi_1 + \varepsilon s(\Phi_1)]) \]

(12)

Finally, using explicit representation of \( s(\Phi_{1,2}) \) and making the differentiation, Eq. (12) can be reduced to

\[
H_- \approx H_- (\Phi_2) - H_- (\Phi_1)
\]

(13)
where
\[
H_-(\Phi) = \frac{1}{4\pi} \tilde{U} (\Phi, s(\Phi)) \left[ \frac{1}{\tau - z - \Phi} \left( -\frac{\partial \Phi}{\partial \rho} \right) + \varepsilon \frac{2\rho}{(\tau - z - \Phi)^2 - \varepsilon \rho^2} \right] \exp (iK [\Phi + \varepsilon s(\Phi)])
\] (14)

Explicit relations for the magnetic field strength differ for above-mentioned cases (i)-(iv). Relationships for the limits of integration \( (\Phi_1^{(i)}, \Phi_2^{(i)}) - (\Phi_1^{(iv)}, \Phi_2^{(iv)}) \) do not depend on the modulation factor. In the general form they were obtained in [8] and in particular form, related to the variable \( \xi = \tau - \sqrt{\rho^2 + (z - z')^2} - z' \), in [5].

The same scheme can be applied to the case of the \( M_+ \) modulation (the analog of the auxiliary integration variable \( \xi \) is \( \zeta = \rho^2 / (z' - z - \sqrt{\rho^2 - (z - z')^2}) \) yielding approximations for corresponding wavefunction \( \Psi_+ \) and the magnetic field strength \( H_+ \).

**General results**

The results obtained by application of the solution scheme to the case of the source current (1)-(3) can be summarized as follows:

- **Case (i)** \( H_+^{(i)} = H_0^{(\pm)} - H_\beta^{(\pm)} \)
- **Case (ii)** \( H_+^{(ii)} = H_T^{(\pm)} - H_\beta^{(\pm)} \)
- **Case (iii)** \( H_+^{(iii)} = H_t^{(\pm)} - H_\beta^{(\pm)} \)
- **Case (iv)** \( H_+^{(iv)} = H_T^{(\pm)} - H_t^{(\pm)} \)

Here the terms
\[
H_\pm^{(\pm)} = \frac{U_\lambda}{4\pi r_\lambda} \chi_\lambda^{(\pm)} R_\chi^{(\pm)}, \quad \lambda = 0, \beta, T, l
\] (16)

originate from the integration-limit functions – the two terms of Eq. (11) – transformed in accordance with (12) into the contributions to the magnetic filed strength (13). Explicit relations for the factors composing (16) are given by (14):

\[
U_\beta = \tilde{U} \left( \sqrt{\frac{1 - \beta}{1 + \beta}} (\tau_\beta - r_\beta), \frac{1 + \beta}{1 - \beta} (\tau_\beta - r_\beta) \right),
\] (17)

\[
U_0 = U_\beta |_{\beta=0} = \tilde{U} (\tau - r, \tau - r),
\] (18)

\[
U_T = \tilde{U} \left( \sqrt{\frac{1 - \beta}{1 + \beta}} (\tau_T - r_T) + T, \frac{1 + \beta}{1 - \beta} (\tau_T - r_T) + T \right),
\] (19)

\[
U_l = \tilde{U} (\tau - r_l - l, \tau - r_l + l),
\] (20)

\[
r_\beta = \sqrt{\rho^2 + z_\beta^2}, \quad z_\beta = \frac{z - \beta \tau}{\sqrt{1 - \beta^2}}, \quad \tau_\beta = \frac{\tau - \beta z}{\sqrt{1 - \beta^2}},
\] (21)
\[ r_0 = r = \sqrt{\rho^2 + z^2}, \quad (22) \]
\[ r_T = \sqrt{\rho^2 + z_T^2}, \quad z_T = \frac{z - \beta (\tau - T)}{\sqrt{1 - \beta^2}}, \quad \tau_T = \frac{\tau - T - \beta z}{\sqrt{1 - \beta^2}}, \quad (23) \]
\[ r_l = \sqrt{\rho^2 + z_l^2}, \quad z_l = z - l, \quad (24) \]
\[ \chi_0^{(\pm)} = \exp \left( \frac{i}{c} \omega_0 (\tau - r) \right), \quad (25) \]
\[ \chi_{\beta}^{(\pm)} = \exp \left( \frac{i}{c} \left[ \omega_0 \left( 1 \pm \frac{\beta}{\Omega} \right) \right] \frac{\tau_{\beta} - r_{\beta}}{\sqrt{1 - \beta^2}} \right), \quad (26) \]
\[ \chi_T^{(\pm)} = \exp \left( \frac{i}{c} \omega_0 T \right) \exp \left( \frac{i}{c} \left[ \omega_0 \left( 1 \pm \frac{\beta}{\Omega} \right) \right] \frac{\tau_T - r_T}{\sqrt{1 - \beta^2}} \right), \quad (27) \]
\[ \chi_l^{(\pm)} = \exp (\pm ik l) \exp \left( \frac{i}{c} \omega_0 (\tau - r_l) \right) \quad (28) \]

and
\[
R_{\lambda}^{(\pm)} = \begin{cases} 
\frac{\Omega \sin \theta_{\lambda}}{1 \pm \Omega \cos \theta_{\lambda}} & \lambda = 0, l \\
\frac{(\Omega \pm \beta) \sin \theta_{\lambda}}{1 \pm \Omega \beta \pm (\Omega \pm \beta) \cos \theta_{\lambda}} & \lambda = \beta, T.
\end{cases} \quad (29)
\]

Here the angles \( \theta_{\lambda} \) correspond to the spheric-coordinate representations

\[ \rho = r_{\lambda} \sin \theta_{\lambda}, \quad z_{\lambda} = r_{\lambda} \cos \theta_{\lambda}, \quad \lambda = 0, \beta, T, l \quad (30) \]

and \( \omega_0 = k v_{\text{phase}} = k \Omega c \) is the source modulation frequency.

**Directionality of the emanated waves**

Inherent directionality of the waves produced by the current pulse with high-frequency filling in question is defined by the terms of the type \( \frac{\Omega \sin \theta_{\lambda}}{1 \pm \Omega \cos \theta_{\lambda}} \) in the case of \( H_0^{(\pm)}, H_l^{(\pm)} \) and those of the type \( \frac{(\Omega \pm \beta) \sin \theta_{\lambda}}{1 \pm \Omega \beta \pm (\Omega \pm \beta) \cos \theta_{\lambda}} \) in the case of \( H_{\beta}^{(\pm)}, H_T^{(\pm)} \).

**Luminal phase velocity**

A special case of the luminal phase velocity, \( v_{\text{phase}} = c, \Omega = 1 \) was investigated in [5] and here is harnessed as a basis for further analysis. It is easily seen that for \( \Omega = 1 \) the factors \( R_{\lambda}^{(\pm)} \) get much simpler forms

\[ R_{\lambda}^{(-)} = \frac{\sin \theta_{\lambda}}{1 - \cos \theta_{\lambda}} = \cot \frac{\theta_{\lambda}}{2}, \quad R_{\lambda}^{(+)} = \frac{\sin \theta_{\lambda}}{1 + \cos \theta_{\lambda}} = \tan \frac{\theta_{\lambda}}{2}, \quad \lambda = 0, l, \beta, T. \quad (31) \]

As shown in [5], radiation due to \( M_- \) modulation is directed along the current-pulse propagation, \( \theta_{\lambda} \simeq 0 \), while the radiation directionality for \( M_+ \) is opposite, \( \theta_{\lambda} \simeq \pi \).
In spite of the apparent divergence of the terms due to the presence of the tangent/cotangent factors, their sums composing the magnetic field strength remains finite everywhere except for the source domain $z \in (0, \beta \tau)$, $\rho = 0$ ($\theta = 0$, $\theta_\beta = \pi$). For example, in the case (ii) of a short current pulse the radiation intensity $J_\pm^{(ii)}$ in the far zone is given by

$$J_\pm^{(ii)} \approx J_0 \left( \frac{U_\beta}{r_\beta} \right)^2 \cot^2 \frac{\theta_\beta}{2} \sin^2 \left( kT_\pm \frac{\beta}{1 + \beta} \sin^2 \frac{\theta_\beta}{2} \right)$$

while in the case (iii), corresponding to a long current pulse we have

$$J_\pm^{(iii)} \approx J_0 \left( \frac{U_\beta}{r_\beta} \right)^2 \tan^2 \frac{\theta_\beta}{2} \sin^2 \left( kT_\pm \frac{\beta}{1 + \beta} \cos^2 \frac{\theta_\beta}{2} \right)$$

where $J_0 = (8\pi^2 c_0)^{-1}$, $\epsilon$ is the permittivity of the medium.

**Cases of subluminal and superluminal phase velocity**

In the case of subluminal and superluminal phase velocity, $\Omega \neq 1$, the angular factors of the field components $R_{\theta_\lambda}^{(\pm)}$ have more complicated forms defined by Eq. (29), depending on one ($\Omega$, cases $\lambda = 0, l$) or two ($\Omega$ and $\beta$, cases $\lambda = \beta, T$) additional parameters. Dependence of the angular factors $R_{0, l}^{(\pm)}$ on $\Omega$ and the angular parameter $\theta_{0, l}$ is illustrated in Fig. 1. As it is seen from the figure, for subluminal phase velocity the tendency of $H_\lambda^{(\pm)}$ to be directed at $\theta_{0, l} = 0$ and $R_{0, l}^{(\pm)}$ at $\theta_{0, l} = \pi$, clearly manifested at $\Omega = 1$, holds down to $\Omega \approx 0.7$. For superluminal phase velocities, the factors $R_{0, l}^{(\pm)}$ demonstrate lateral (i.e., towards $\theta_{0, l} = \pi/2$) shift of the propagation directionality, which is observed in the vicinity of the $R_{0, l}^{(\pm)}$ singularity curves, $\theta_{0, l} = \arccos \Omega^{-1}$ for $M_-$ and $\theta_{0, l} = \pi - \arccos \Omega^{-1}$ for $M_+$. For analysis of more complicated factors $R_{0, l}^{(\pm)}$ let us represent them in a two-parameter form

$$R_{\beta, T}^{(\pm)} = \frac{\sin \theta_{T, T}}{\Theta^{(\pm)} \pm \cos \theta_{T, T}}, \quad \Theta^{(\pm)} = \frac{1 \pm \Omega \beta}{\Omega \pm \beta}$$

As functions of $\beta$ and $\Omega$, the parameters $\Theta^{(-)}$ and $\Theta^{(+)}$ demonstrate different behavior: $\Theta^{(-)}$ varies from $-\infty$ to $-\infty$, having the area of singularity $\Omega = \beta$ while the sign of $\Theta^{(+)}$ is always positive and the area of its singularity is limited to a single point $\Omega = \beta = 0$, see Fig. 2. Note that $\lim_{\beta \to 1} \Theta^{(\pm)} = \pm 1$, $\lim_{\Omega \to +\infty} \Theta^{(\pm)} = \pm \beta$, and $\Theta^{(\pm)} \big|_{\Omega=1} = 1$. 
Figure 1: Dependence of the factors $R_{0,l}^{(±)}$ on the dimensionless modulation phase velocity $Ω$ and the angular parameter $θ_{0,l}$.

The angular factors $R_{0,l}^{(±)}$ are finally plotted as functions of $Θ^{(±)}$ and $θ_{0,l}$ in Fig. 3. It is easily seen that $R_{0,l}^{(−)}$ imply directionality of the corresponding terms $H_{β,T}^{(−)}$ at

$$\cos θ_{β,T} \simeq Θ^{(−)}$$

which implies $−1 < Θ^{(−)} < 1$. As follows from (34), the latter condition is achieved provided that $Ω > 1$, that is, only for luminal and superluminal phase velocities. As $Ω$ varies from 1 to $+∞$, the parameter $Θ^{(−)}$ runs through the values from 1 to $−β$, providing directionality of $H_{β,T}^{(−)}$ in the range $0 < θ_{β,T} < π/2 + \arcsin β$. For the phase velocity close to the velocity of light, when $0 < \frac{1+β}{1−β}(Ω − 1) << 1$, the directionality
Figure 2: Parameters $\Theta^{(\pm)}$ plotted versus the dimensionless modulation phase velocity $\Omega$ and the dimensionless source-current front velocity $\beta$.

Figure 3: The angular factors $R^{(\pm)}_{\beta,T}$ plotted as functions of $\Theta^{(\pm)}$ and $\theta_{\beta,T}$.

of the magnetic field components $H^{(-)}_{\beta,T}$ is observed for

$$\max \left( H^{(-)}_{\beta,T} \right) : \quad \theta_{\beta,T} \simeq \sqrt{\frac{1 + \beta}{1 - \beta}} (\Omega - 1)$$

The factors $R^{(+)}_{\beta,T}$ provide field directionality in the area $0 < \Theta^{(+)} < 1$ (achieved for $\theta_{\beta,T} \simeq \pi - \arccos (\Theta^{(+)}$) as well only in the case of luminal/superluminal phase velocities. The localization direction lies in the range $\pi/2 + \arcsin (\beta < \theta_{\beta,T} < \pi$ and for the case $0 < \frac{1 - \beta}{1 + \beta} (\Omega - 1) << 1$ the $H^{(+)}_{\beta,T}$ directionality can be estimated as

$$\max \left( H^{(+)}_{\beta,T} \right) : \quad \theta_{\beta,T} \simeq \pi - \sqrt{\frac{1 - \beta}{1 + \beta}} (\Omega - 1) .$$
**FREQUENCY TRANSFORM**

The modulation frequencies corresponding to the terms $H^{(\pm)}_{0,\beta,T}$ composing the magnetic field strength are defined by the arguments of the exponential factors $\chi^{(\pm)}_{0,\beta,T,T}$. The factors $\chi^{(\pm)}_0 = \exp\left(\frac{i}{c}\omega_0 (\tau - r)\right)$ and $\chi^{(\pm)}_i = \exp(\pm i kl)\exp\left(\frac{i}{c}\omega_0 (\tau - r_i)\right)$ correspond to the initial modulation frequency $\omega_0$, so the frequency transform is observed only in $H^{(\pm)}_{\beta,T}$.

The frequency transformation ranges can be found considering wave propagation in the two limiting directions: parallel ($\theta_{\beta,T} = 0$) and anti-parallel ($\theta_{\beta,T} = \pi$) to the direction of propagation of the source current. In the case $\theta_{\beta,T} = 0$ one has $\tau_{\beta,T} - r_{\beta,T} = \tau_{\beta,T} - z_{\beta,T}$. Coming back to the initial frame $\tau, z$, we get

$$\chi^{(\pm)}_{\beta,T} = const \times \exp\left(\frac{i}{c} \left[\omega_0 \frac{|1 - \beta/\Omega|}{1 - \beta}\right] (\tau - r)\right), \quad \cos \theta_{\beta,T} = 1 \quad (38)$$

The same consideration for the case $\theta_{\beta,T} = \pi \Rightarrow \tau_{\beta,T} - r_{\beta,T} = \tau_{\beta,T} + z_{\beta,T}$ leads us to

$$\chi^{(\pm)}_{\beta,T} = const \times \exp\left(\frac{i}{c} \left[\omega_0 \frac{|1 + \beta/\Omega|}{1 + \beta}\right] (\tau - r)\right), \quad \cos \theta_{\beta,T} = -1 \quad (39)$$

Thus for different directions of wave propagation $\theta$ the contributions $H^{(\pm)}_{\beta,T}$ to the total magnetic field are subjected to frequency transformation with respect to the initial modulation frequency of the source current $\omega_0$:

$$\omega_0 \rightarrow \omega^{(\pm)}_{\beta,T} (\theta).$$

The range of this transformation is defined by the following inequality

$$\omega^{(\pm)}_{\beta,T} (\pi) = \omega_0 \frac{|1 + \beta/\Omega|}{1 + \beta} \leq \omega^{(\pm)}_{\beta,T} (\theta) \leq \omega^{(\pm)}_{\beta,T} (0) = \omega_0 \frac{|1 - \beta/\Omega|}{1 - \beta}. \quad (40)$$

In the case of a nearly luminal phase velocity, $|\Omega - 1| << 1$, one can observe manifestation of phenomena similar to those described in [5] for the particular case $\Omega = 1$ : the red shift

$$\omega_0 \left[\frac{1 - \beta}{1 + \beta} + (\Omega - 1) \frac{\beta}{1 + \beta}\right] \leq \omega^{(\pm)}_{\beta,T} (\theta) \leq \omega_0 \left[1 + (\Omega - 1) \frac{\beta}{1 - \beta}\right]. \quad (41)$$

for $M_-$ modulation and the ultraviolet shift

$$\omega_0 \left[1 - (\Omega - 1) \frac{\beta}{1 + \beta}\right] \leq \omega^{(\pm)}_{\beta,T} (\theta) \leq \omega_0 \left[1 + \beta - (\Omega - 1) \frac{\beta}{1 - \beta}\right]. \quad (42)$$

for $M_+$ modulation.

**Beatings**

As discussed in [5], the cases (i) and (iv) correspond to excitation of waves of two different frequencies, which in some conditions leads to appearance of the low-frequency components (beatings), corresponding to frequency subtraction $\Omega_\Delta \approx \mid \omega_0 - \omega^{(\pm)}_{\beta,T} \mid$ at
mixing $H^{(\pm)}_{0,l}$ with $H^{(\pm)}_{3,T}$. Remarkably, in the case $\Omega < 1$ one more type of beatings is observed for $\Omega \simeq \beta$ in a single component $H^{(-)}_{3,T}$ due to subtraction of the velocities $\Omega$ and $\beta$ in (40): $\omega^{(-)}_{3,T}(\theta) \to 0$ as $\Omega \to \beta$.

**Conclusion**

For arbitrary slowly varying envelope of the current pulse, a closed-form quadrature expression is obtained for the magnetic component of the field, which enables the entire field to be reconstructed at long distances from the source. Separation of the modulating factor from the envelope enables one to illustrate and analytically describe the following phenomena:

- Directionality of the emanated waves. As in the case of the luminal phase modulation velocity described in [5], this directionality results from the interference of the partial waves emanated by a finite-length ($T$) current pulse emitted from one end and absorbed at the other end of a finite-length ($l$) radiator. In the case of $\Omega \neq 1$ the phase factors are more intricate and the interference nature is more complicated than for $\Omega = 0$. As so, the directionality angles reside in the direction different from those of source propagation and counterpropagation.

- Transformation of the frequency of the electromagnetic wave carrier with respect to the initial frequency of the source current modulation, which takes place for certain wave regimes and manifests itself as the red or ultraviolet shift for the modulation factors $M_-$ and $M_+$ correspondingly.

- In certain space-time domains, the waves of two different frequencies, fundamental and shifted, are excited, which leads to formation of beating-type interferential patterns.

- One more type of beatings can be observed in the case of $M_-$ modulation when the phase modulation velocity comes close to the source-current pulse propagation velocity.

As far as in the intermediate stage the solution scheme reduces the electromagnetic problem to a scalar problem (5) containing the wave equation, results obtained may be readily generalized to the case of scalar waves, the wave process being completely described by the wavefunction $\Psi$ in both near and far zones.

**References**


