Preface

The International Conferences “Days on Diffraction” are annually held by the Faculty of Physics of St. Petersburg University, St. Petersburg Branch of V.A. Steklov Mathematical Institute and Euler International Mathematical Institute of the Russian Academy of Sciences.

Approximately 140 scientists from all over the world took part in the "Days on Diffraction - 2009" Conference. The Organizing Committee is thankful to all the participants. We appreciate their presentations which have been made during plenary, parallel and poster sessions. Our special gratitude is to the authors of 36 papers selected for publication in the Proceedings for preparation of their manuscripts in accordance with the required rules.

The Organizing Committee

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This year we celebrated the 80-th anniversary of Vladimir Sergeevich Buldyrev – the eminent scientist in the field of mathematical theory of wave phenomena! The progress in science would be impossible without scientific schools which in its term need “condensation centers”. As an example professor G. I. Petrashen could be nominated as such center in the 50-ies. Later on this role had moved to younger researches among whom was Vladimir Sergeevitch Buldyrev. Vladimir Sergeevitch had been a remarkable organizer of science: more than twenty of his pupils got their doctoral degree and six of them became professors, the initiation of “Diffraction Days” is strongly binded with his name.

He was the author and coauthor of many pioneering articles and books of mathematical theory of diffraction. He had been awarded by State Prize and St.-Petersburg university Prize

Study of “whispering gallery” and “jumping ball” modes led Vladimir Sergeevitch to the notion of stability of rays in the first approximation and to elaboration of the so-called “infinitesimal ray method”. Together with boundary layer approach originated by V. A. Fock these studies open a new epoch in diffraction theory which is continuing up to nowadays. In the papers of V. S. Buldyrev his “analytical might” is striking. He wrote four textbooks and monographs. Most of them appeared in English and Russian. They are well-known by students and scientists.

Beyond scientific activity Vladimir Sergeevitch with his wife Aida Andreevna and with his friends climbed many mountains, paddled down several rivers and travelled in many winter ski tours.

We wish Vladimir Sergeevitch new successes and mainly good health.

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Undistorted progressive waves resulting from separation of variables in some orthogonal curvilinear coordinate systems

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New families of waves, solutions of the homogeneous wave equation akin to Bateman-Hillion relatively undistorted progressive waves and focus waves, where constructed on the basis of separation of variables in the wave equation and specific Bateman transformations in the circular cylindrical, elliptic cylindrical, and parabolic cylindrical coordinate systems.

1 Introduction

The Bateman transformation [1]

\[ \Psi_C(x, y, z, \tau) \rightarrow \tilde{\Psi}_C(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\tau}) = \frac{1}{\tau - z} \Psi_C(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\tau}) \] (1)

allows a new solution of the homogeneous wave equation \( \Psi_C \) to be generated on the basis of some known solution \( \tilde{\Psi}_C \). Here \( x, y, z \) and \( \tau \) stand for the Cartesian coordinates and time \( t \), represented in a dimensionless form, \( x/\lambda \rightarrow x, y/\lambda \rightarrow y, z/\lambda \rightarrow z, ct/\lambda \rightarrow \tau \), using some characteristic length \( \lambda \) and the wavefront velocity \( c \), the transformed coordinates being

\[ \begin{align*}
   \tilde{x} &= \frac{x}{\tau - z}, \\
   \tilde{y} &= \frac{y}{\tau - z}, \\
   \tilde{z} &= \frac{x^2 + y^2 + z^2 - \tau^2 - 1}{2(\tau - z)}, \\
   \tilde{\tau} &= \frac{x^2 + y^2 + z^2 - \tau^2 + 1}{2(\tau - z)}.
\end{align*} \] (2)

One can employ (1) for generating relatively undistorted progressive waves [2, 3]

\[ \Psi(x, y, z, \tau) = g(x, y, z, \tau) f(\Phi(x, y, z, \tau)) \]

with the phase function

\[ \Phi(x, y, z, \tau) = z + \tau + \frac{x^2 + y^2}{\tau - z} = \tilde{z} + \tilde{\tau} \]

characteristic for the Bateman solutions [1, 3] that were in the origin of many interesting wavefunctions, in particular, Brittingham’s focus wave modes and Bateman-Hillion relatively undistorted progressive waves [4].

In [5], initial solutions \( \Psi \) were constructed via separation of variables in the homogeneous wave equation with zero separation constant. Several orthogonal coordinate systems \( x_1, x_2, x_3 \) of the cylindrical type

\[ x = X(x_1, x_2), \quad y = Y(x_1, x_2), \quad z = x_3 \] (3)

were considered: Cartesian (rectangular), circular (ordinary) cylindrical, elliptic cylindrical, parabolic cylindrical and bipolar. Borisov [6] extended this technique to the case of a non-zero separation constant \( a^2 \), obtaining new propagation-invariant wave structures in the circular cylindrical \( (x_1 = \rho, x_2 = \varphi, x_3 = z) \) and spherical \( (x_1 = r, x_2 = \varphi, x_3 = \theta) \) coordinate systems, where transformation (1) is invariant with respect to the polar angle: \( \tilde{x}_2 = x_2 (\tilde{\varphi} = \varphi) \). In the present work the latter approach is applied to the solutions of the wave equations constructed using two other cylindrical orthogonal coordinates (3) admitting separation of variables in the resulting 2D Helmholtz equation.

2 General relations

First let us pass to the orthogonal cylindrical coordinates (3) and represent the initial wavefunction

\[ \Psi(x_1, x_2, z, \tau) = \Psi_C(X(x_1, x_2), Y(x_1, x_2), z, \tau) \]

in the form

\[ \Psi(x_1, x_2, z, \tau) = w(x_1, x_2) v(z, \tau) \] (4)

Substituting (4) into the homogeneous wave equation, which in the particular case of cylindrical coordinates, \( x_3 = z, h_{1,2} = h_{1,2}(x_1, x_2), h_3 = 1 \) (\( h_{1,2,3} \) are the metric coefficients), takes the form

\[ \frac{1}{h_1 h_2} \left( \frac{\partial}{\partial x_1} \left( h_2 \frac{\partial \Psi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_1 \frac{\partial \Psi}{\partial x_2} \right) \right) + \frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \Psi}{\partial \tau^2} = 0 \]
one has
\[ \frac{1}{h_1 h_2} \left( \frac{\partial}{\partial x_1} \left( h_2 \frac{\partial w}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_1 \frac{\partial w}{\partial x_2} \right) \right) v + \left( \frac{\partial^2 v}{\partial^2 x_1} - \frac{\partial^2 v}{\partial^2 x_2} \right) w = 0 \] (5)

Separation of variables with the separation constant \( a^2 \) splits (5) into the 1D Klein-Gordon equation
\[ \frac{\partial^2 v}{\partial^2 x_1} - \frac{\partial^2 v}{\partial^2 x_2} + a^2 v = 0 \]
which can be satisfied by an exact analytical solution [6]
\[ v_{ak} (z, \tau) = \exp \left( i \left( k\tau + \sqrt{k^2 - a^2} z \right) \right) \] (6)
and the 2D Helmholtz equation
\[ \frac{1}{h_1 h_2} \left( \frac{\partial}{\partial x_1} \left( h_2 \frac{\partial w}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( h_1 \frac{\partial w}{\partial x_2} \right) \right) + a^2 w = 0 \] (7)

The solvability of the latter equation implies restrictions on the coordinate systems in which we can generate desired wavefunctions. The Helmholtz differential equation admits solution by separation of variables
\[ w_u (x_1, x_2) = w_{u_1}(x_1) w_{u_2}(x_2) \] (8)
(where the functions \( w_{u_1,2} \) acquire a parametric dependence on an additional constant \( \mu \) resulting from \( x_1-x_2 \) separation) in eleven orthogonal coordinate systems [7], from which three cylindrical coordinates – ordinary, elliptic and parabolic – are of required type (3). Construction of undistorted progressive waves in the (ordinary) cylindrical coordinate system is discussed in [6] and will be presented here only for the sake of consistency.

The desired form of the Bateman transformation (1) in curvilinear coordinates
\[ \Psi (x_1, x_2, z, \tau) \rightarrow \tilde{\Psi} (\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{\tau}) \] (9)
is governed by equations
\[ \Psi (x_1, x_2, z, \tau) = \Psi_C (X (x_1, x_2), Y (x_1, x_2), z, \tau), \]
\[ \tilde{\Psi} (\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{\tau}) = \tilde{\Psi}_C (X (\tilde{x}_1, \tilde{x}_2), Y (\tilde{x}_1, \tilde{x}_2), \tilde{z}, \tilde{\tau}) \]
and notion (9) takes the form
\[ \Psi (x_1, x_2, z, \tau) \rightarrow \tilde{\Psi} (\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{\tau}) = \frac{1}{z - \tau} \tilde{\Psi} (\tilde{x}_1, \tilde{x}_2, \tilde{z}, \tilde{\tau}) \]
where the variables \( \tilde{z} \) and \( \tilde{\tau} \) are defined explicitly,
\[ \tilde{z}, \tilde{\tau} = \frac{X^2 (x_1, x_2) + Y^2 (x_1, x_2) + z^2 - r^2 + 1}{2(z - \tau)} \]
while \( \tilde{x}_1 \) and \( \tilde{x}_2 \) obey the implicit relations that follow from (2)
\[ \frac{X (\tilde{x}_1, \tilde{x}_2)}{Y (\tilde{x}_1, \tilde{x}_2)} = \frac{1}{z - \tau} \frac{X (x_1, x_2)}{Y (x_1, x_2)} \] (10)

In principle, (10) may result in more than one solution, giving rise to several different curvilinear-coordinate representations of the same cartesian-coordinate Bateman transformation (1).

Applying the Bateman transformation (9) to the solution of the wave equation resulted from (6) and (8),
\[ \Psi_{u_{1,2}} (x_1, x_2, z, \tau) = v_{ak} (z, \tau) w_{u_{1,2}} (x_1) w_{u_{1,2}} (x_2) \]
we have
\[ \tilde{\Psi}_{u_{1,2}} (x_1, x_2, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak} (\tilde{z}, \tilde{\tau}) \times w_{u_{1,2}} (\tilde{x}_1, x_2, z, \tau) \] (11)
where
\[ \tilde{v}_{ak} (\rho, z, \tau) = v_{ak} (\tilde{z}, \tilde{\tau}) = \exp \left( \frac{i}{2} \left( k + \sqrt{k^2 - a^2} \right) \right) \]
\[ \times \left( z + \tau - \frac{1}{z - \tau} \frac{r^2}{k + \sqrt{k^2 - a^2}} \right) \]
\[ \rho = \sqrt{\rho_x^2 + \rho_y^2} = \sqrt{X^2 (x_1, x_2) + Y^2 (x_1, x_2)} \]
and the transformed coordinates \( \tilde{x}_1, \tilde{x}_2 \) are defined by (10).

Note that (11) is a generalization of the propagation-invariant space-time structure, reported earlier for the circular cylindrical coordinates [6], Eq. (3)).

3 Particular cases

3.1 Circular cylindrical coordinates

This case was thoroughly investigated in [6] and is given here as an illustrative example of how the general relations obtained in the previous section can be concretized, readily yielding consistent results. In the case in question \( x_1 \equiv \rho, x_2 \equiv \varphi, X (x_1, x_2) = X (\rho, \varphi) = \rho \cos \varphi, Y (x_1, x_2) = Y (\rho, \varphi) = \rho \sin \varphi, \) and (10) yields
\[ \tilde{\rho} = \rho/ (z - \tau), \quad \tilde{\varphi} = \varphi \]
\[ \tilde{\varphi} = \varphi + \pi \] (12)
Correspondingly, the Bateman transformation takes two forms, yielding
\begin{equation}
\tilde{\Psi}^{(1)} = \frac{1}{z - \tau} \Psi \left( \frac{\rho}{z - \tau}, \varphi, \tilde{z}, \tilde{\tau} \right)
\end{equation}
\begin{equation}
\tilde{\Psi}^{(2)} = \frac{1}{z - \tau} \Psi \left( \frac{\rho}{z - \tau}, \varphi + \pi, \tilde{z}, \tilde{\tau} \right)
\end{equation}

The Helmholtz equation (7) is solved with
\begin{equation}
w_{\mu 1}(x_1) = w_{\mu 1}(\rho) = Z_\mu (a\rho),
\end{equation}
\begin{equation}
w_{\mu 2}(x_2) = w_{\mu 2}(\varphi) = \exp (i\mu\varphi)
\end{equation}
and (11) results in two progressive waves
\begin{equation}
\tilde{\Psi}_{\mu k}(\rho, \varphi, \tilde{z}, \tilde{\tau}) = \frac{1}{z - \tau} \tilde{v}_{uk}(\rho, \tilde{z}, \tilde{\tau})
\end{equation}
\begin{equation}
\times Z_\mu \left( \frac{a\rho}{z - \tau} \right) \exp (i\mu\varphi)
\end{equation}
where $Z_\mu$ is a solution of the Bessel equation; $\tilde{v}_{uk}$ results from (14) after dropping a constant factor $-\exp (i\mu\pi)$.

### 3.2 Parabolic cylindrical coordinates

The parabolic cylindrical coordinate system, in which $x_1 \equiv \xi, x_2 \equiv \eta, h\xi = h\eta = \sqrt{\xi^2 + \eta^2},$
\begin{equation}
x = X(\xi, \eta) = \frac{1}{2} (\xi^2 - \eta^2), \quad y = Y(\xi, \eta) = \xi\eta
\end{equation}
results in Eq. (10) in the form
\begin{equation}
\tilde{\xi}^2 - \tilde{\eta}^2 = \frac{1}{z - \tau} (\xi^2 - \eta^2), \quad \tilde{\xi}\tilde{\eta} = \frac{1}{z - \tau} \xi\eta
\end{equation}
which yields immediately four possible coordinate representations
\begin{equation}
\begin{cases}
\tilde{\xi} = \xi/\sqrt{z - \tau} \\
\tilde{\eta} = \eta/\sqrt{z - \tau}
\end{cases}
\quad \begin{cases}
\tilde{\xi} = -\xi/\sqrt{z - \tau} \\
\tilde{\eta} = -\eta/\sqrt{z - \tau}
\end{cases}
\quad \begin{cases}
\tilde{\xi} = \eta/\sqrt{z - \tau} \\
\tilde{\eta} = -\xi/\sqrt{z - \tau}
\end{cases}
\quad \begin{cases}
\tilde{\xi} = -\eta/\sqrt{z - \tau} \\
\tilde{\eta} = \xi/\sqrt{z - \tau}
\end{cases}
\end{equation}
The Helmholtz equation (7) turns into
\begin{equation}
\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + a^2 (\xi^2 + \eta^2) w = 0
\end{equation}
which, being separated in accordance with (8), yields
\begin{equation}
\frac{d^2 w_{\mu 1}}{d\xi^2} - (\mu - a^2\xi^2) w_{\mu 1} = 0,
\end{equation}
\begin{equation}
\frac{d^2 w_{\mu 2}}{d\eta^2} + (\mu + a^2\eta^2) w_{\mu 2} = 0
\end{equation}
Both equations are particular forms of the (generalized) parabolic cylinder equation [8]. After applying “Procedure 2” proposed by Zwillinger ([9], p. 129) and presenting (16) in the canonical form examined in [10], p. 686, one can express the desired solutions in terms of the parabolic cylinder functions $D_\nu (Z)$ (see also [7])
\begin{equation}
w_{\mu 1}(\xi) = c_{11} D_{\frac{\sqrt{\tau - z}}{\sqrt{\tau - z}} - c_{12} D_{\frac{\sqrt{\tau - z}}{\sqrt{\tau - z}}} - c_{12} D_{\frac{\sqrt{\tau - z}}{\sqrt{\tau - z}}},
\end{equation}
\begin{equation}
w_{\mu 2}(\eta) = c_{21} D_{\frac{\sqrt{\tau - z}}{\sqrt{\tau - z}}} + c_{22} D_{\frac{\sqrt{\tau - z}}{\sqrt{\tau - z}}},
\end{equation}
where $c_{11}, c_{12}, c_{21}, \text{and } c_{22}$ stand for arbitrary (in general, complex) constant values. Noting that
\begin{equation}
\rho = \sqrt{X^2(\xi, \eta) + Y^2(\xi, \eta)} = \frac{1}{2} (\xi^2 + \eta^2)
\end{equation}
one gets four different explicit representations of the constructed parabolic cylinder progressive waves
\begin{equation}
\tilde{\psi}^{(1)}(\xi, \eta, \tau) = \frac{1}{z - \tau} \tilde{v}_{uk}(\rho, \tau)
\end{equation}
\begin{equation}
\times w_{\mu 1} \left( \frac{\pm \xi}{\sqrt{\tau - z}} \right) w_{\mu 2} \left( \frac{\pm \eta}{\sqrt{\tau - z}} \right),
\end{equation}
\begin{equation}
\tilde{\psi}^{(2)}(\xi, \eta, \tau) = \frac{1}{z - \tau} \tilde{v}_{uk}(\rho, \tau)
\end{equation}
\begin{equation}
\times w_{\mu 1} \left( \frac{\pm \eta}{\sqrt{\tau - z}} \right) w_{\mu 2} \left( \frac{\pm \xi}{\sqrt{\tau - z}} \right).
\end{equation}

### 3.3 Elliptic cylindrical coordinates

In the elliptic cylindrical coordinate system, for which $x_1 \equiv u$ and $x_2 \equiv \phi$, Eq. (3) is concretized in
\begin{equation}
x = X(u, \phi) = h \cosh u \cos \phi,
\end{equation}
\begin{equation}
y = Y(u, \phi) = h \sinh u \sin \phi,
\end{equation}
where $h$ is a real positive parameter characterizing the common semidistance of confocal ellipses with eccentricities $1/ \cosh u$ and confocal hyperbolas with eccentricities $1/ \cos \phi$. Turning to the complex variable $\phi - iu$, one gets (10) in the form
\begin{equation}
\cos (\phi - iu) = \frac{1}{2i} \cos (\phi - iu)
\end{equation}
which results in two independent solutions
\begin{equation}
\begin{cases}
\hat{a} = -\text{Im} \left( \arccos \left( \frac{1}{2i} \cos (\phi - iu) \right) \right) \\
\hat{\phi} = \text{Re} \left( \arccos \left( \frac{1}{2i} \cos (\phi - iu) \right) \right)
\end{cases}
\end{equation}
\[
\begin{align*}
\begin{cases}
\tilde{u} &= -\text{Im} \left( \arccos \left( \frac{1}{z} \cos (\phi - iu) \right) \right) \\
\tilde{\phi} &= \text{Re} \left( \arccos \left( \frac{1}{z} \cos (\phi - iu) \right) \right) + \pi
\end{cases}
\end{align*}
\]
(18)

where the value of the multivalued complex arc cosine function is fixed in such a way that
\[
\lim_{u \to \infty} \text{Re} \left( \arccos \left( \Lambda \cos (\phi - iu) \right) \right)
= \lim_{u \to \infty} \text{Im} \left( \log \left( \Lambda \cos (\phi - iu) \right) \right) + \pi
= \lim_{u \to \infty} \text{Im} \left( \log \left( \Lambda \exp (\phi + u) \right) \right) = \phi
\]
for any real positive parameter \( \Lambda \), maintaining unambiguity of the dependence \( \tilde{\phi} = \tilde{\phi}(\phi, u) \) for all values of \( \phi \) and making formulas (17), (18) analogous to relations (29) established for the circular cylindrical coordinates. Consequently,
\[
\lim_{u \to \infty} \tilde{u} = u \log |\Lambda|, \quad \Lambda = \pm \frac{1}{z - \tau}
\]

Being a part of the parametric representation of the straight line \( \tilde{x} = \Lambda x, \tilde{y} = \Lambda y, \Lambda = \frac{1}{z - \tau} \), the equations for \( \tilde{\phi} \) in systems (17) and (18) reduce, for \( y = 0 \), into their counterparts in the circular cylindrical coordinates
\[
\tilde{\phi}(\phi, u) = \begin{cases}
\phi + \pi, & \phi = 0, \pm \pi, \pm 2\pi, ...
\end{cases}
\]
(20)

For the former case, \( \Lambda = 0.5 \), and \( u = 1.4 \), this relation is illustrated in Fig. 1.

The Bateman transformation takes the following forms
\[
\tilde{\Psi}^{(1)}(u, \phi, z, \tau) = \frac{1}{z - \tau} \Psi \left( -\text{Im} \zeta_+, \text{Re} \zeta_+, \tilde{z}, \tilde{\tau} \right)
\]
(21)
\[
\tilde{\Psi}^{(2)}(u, \phi, z, \tau) = \frac{1}{z - \tau} \Psi \left( -\text{Im} \zeta_-, \text{Re} \zeta_- + \pi, \tilde{z}, \tilde{\tau} \right)
\]
(22)

where
\[
\zeta_{\pm} = \arccos \left( \pm \frac{1}{z - \tau} \cos (\phi - iu) \right),
\]
\[
\tilde{z}, \tilde{\tau} = \frac{\rho^2 + z^2 - \tau^2 \mp 1}{2(z - \tau)},
\]
\[
\rho = h \sqrt{\cosh^2 u \cos^2 \phi + \sinh^2 u \sin^2 \phi}
\]
The separation of variables \( u \) and \( \phi \) splits the Helmholtz equation (7) into the system
\[
\begin{align*}
\frac{d^2 w_{a\mu 1}}{du^2} + \left( \mu - \frac{h^2 a^2}{2} - \frac{h^2 a^2}{4} \cosh 2u \right) w_{a\mu 1} &= 0 \\
\frac{d^2 w_{a\mu 2}}{d\phi^2} + \left( \mu - \frac{h^2 a^2}{2} - \frac{h^2 a^2}{4} \cos 2\phi \right) w_{a\mu 2} &= 0
\end{align*}
\]
(23)

that yields \( w_{a\mu 1, 2} \) in terms of the modified \( M(\alpha, q, \phi) \) and ordinary \( M(\alpha, q, \phi) \) Mathieu functions – solutions of the modified and ordinary Mathieu equations [10] – as
\[
\begin{align*}
w_{a\mu 1}(u) &= M \left( \mu - \frac{h^2 a^2}{2}, \frac{h^2 a^2}{4}, u \right) \\
w_{a\mu 2}(\phi) &= M \left( \mu - \frac{h^2 a^2}{2}, \frac{h^2 a^2}{4}, \phi \right)
\end{align*}
\]

Finally, we construct the following progressive waves
\[
\tilde{\Psi}_{a\mu}^{(1)}(u, \phi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak} \left( \rho, z, \tau \right)
\]
\[
\times \tilde{M} \left( \mu - \frac{h^2 a^2}{2}, \frac{h^2 a^2}{4}, -\text{Im} \zeta_+ \right)
\]
\[
\times \tilde{M} \left( \mu - \frac{h^2 a^2}{2}, \frac{h^2 a^2}{4}, \text{Re} \zeta_+ \right)
\]
\[
\tilde{\Psi}_{a\mu}^{(2)}(u, \phi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak} \left( \rho, z, \tau \right)
\]
\[
\times \tilde{M} \left( \mu - \frac{h^2 a^2}{2}, \frac{h^2 a^2}{4}, -\text{Im} \zeta_- \right)
\]
\[
\times \tilde{M} \left( \mu - \frac{h^2 a^2}{2}, \frac{h^2 a^2}{4}, \text{Re} \zeta_- \right)
\]
(23)

(23)

Figure 1: Transformed angular variable \( \tilde{\phi}(\phi, u) \) plotted versus the initial angular variable \( \phi \) for \( \Lambda = 0.5 \) and \( u = 1.4 \).
4 Conclusive note on construction of physically feasible solutions

The complex nature of the obtained wavefunctions \( \Psi \) does not preclude their physical feasibility: it just indicates that both the real and complex parts of the wavefunction separately satisfy the homogeneous wave equation, yielding two real-valued solutions. On the other hand, if one deals with the representation of waves in the absence of specific angular boundary conditions, the solutions must be periodic with respect to the circular coordinate, \( \varphi \) in the case of the circular cylindrical and \( \phi \) in the case of the elliptic cylindrical systems. In the circular cylindrical coordinates this leads to the requirement of \( \mu \) being integer, \( \mu = m = 0, \pm 1, \pm 2, \ldots \). As a consequence, the solutions (15) can be represented as a linear combination of the Bessel \((J_m)\) and Neumann \((N_m)\) functions

\[
\tilde{\Psi}_{ak\mu}^{(1)}(\rho, \varphi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak}(\rho, z, \tau) \times \left( c_{11} J_m \left( \frac{\alpha \rho}{z - \tau} \right) + c_{12} N_m \left( \frac{\alpha \rho}{z - \tau} \right) \right) \exp(i m \varphi),
\]

\[
\tilde{\Psi}_{ak\mu}^{(2)}(\rho, \varphi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak}(\rho, z, \tau) \times \left( c_{21} J_m \left( \frac{\alpha \rho}{\tau - z} \right) + c_{22} N_m \left( \frac{\alpha \rho}{\tau - z} \right) \right) \exp(i m \varphi),
\]

or, alternatively, as a linear combination of the Hankel functions

\[
\tilde{\Psi}_{ak\mu}^{(1)}(\rho, \varphi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak}(\rho, z, \tau) \times \left( c_{11} H^{(1)}_{im} \left( \frac{\alpha \rho}{z - \tau} \right) + c_{12} H^{(2)}_{im} \left( \frac{\alpha \rho}{z - \tau} \right) \right) \exp(i m \varphi),
\]

\[
\tilde{\Psi}_{ak\mu}^{(2)}(\rho, \varphi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak}(\rho, z, \tau) \times \left( c_{21} H^{(1)}_{im} \left( \frac{\alpha \rho}{\tau - z} \right) + c_{22} H^{(2)}_{im} \left( \frac{\alpha \rho}{\tau - z} \right) \right) \exp(i m \varphi),
\]

see [11], Eq. (21.8-8).

Due to fixing of the multivalued complex arc cosine function in accordance to (19), the dependence \( \tilde{\phi} = \tilde{\phi}(\phi, u) \) is not periodic with respect to \( \phi \), yielding instead

\[
\tilde{\phi}(\phi + 2\pi m, u) = \tilde{\phi}(\phi, u) + 2\pi m, \quad m = \pm 1, \pm 2, \ldots
\]

In these circumstances the condition of periodicity of the wavefunctions (23) with respect to the “physical” circular variable \( \tilde{\phi} \) follows from the periodicity of the Mathieu function \( M \) with respect to its last argument. As so, one possible representation of (23) possessing the desired periodicity is

\[
\tilde{\Psi}_{ak\mu}^{(1)}(\rho, \phi, z, \tau) = \frac{1}{z - \tau} \tilde{v}_{ak}(\rho, z, \tau) \times C e_m(\tilde{u}, q) \ c e_m(\tilde{\phi}, q),
\]

\[
\left( \begin{array}{c} \tilde{u} \\ \tilde{\phi} \end{array} \right) = \left( \begin{array}{cc} \text{Re} & -\text{Im} \\ \text{Im} & \text{Re} \end{array} \right) \arccos \left( \frac{1}{z - \tau} \cos(\phi - i u) \right),
\]

\[
q = \frac{\hbar^2}{2} a^2 \frac{4}{m^2}
\]

where \( C e_m \) is the radial Mathieu cosine and \( c e_m \) the angular Mathieu cosine [12], periodic with respect to \( \tilde{\phi} \). Such a wavefunction can be constructed only for one specific value of the separation constant \( \mu = \mu_m \), chosen in such a manner that

\[
\mu_m - \frac{\hbar^2 a^2}{2} = a_m(q)
\]

where \( a_m(q) \) is the \( m \)-th characteristic number related to the angular Mathieu cosine \( c e_m(\tilde{\phi}, q) \). Other possible representations include using the Mathieu sines \( (S e_m, s e_m) \), which together with the Mathieu cosines compose a complete orthogonal system [12], as well as alternative radial functions of the second kind \( N e_m \) (see [13], p. 1411) and several other functions \( (F e_m, G e_m, G y_m, \text{etc.}) \) discussed in [12], Chapter 8. Such solutions represent a family of Mathieu beams essentially different from that introduced by Gutiérrez-Vega et al. [14] for the case of complete separation of the variables \( u, \phi, z, \tau \) in the wave equation.

As shown by McLachlan in Appendix I of Ref. [12], then \( h \to 0 \) and \( u \to \infty \), maintaining \( \rho = \text{const} \), one has

\[
\left( \begin{array}{c} c e_m(\tilde{\phi}, q) \\ s e_m(\tilde{\phi}, q) \end{array} \right) = \left( \begin{array}{c} \cos m \tilde{\phi} \\ \sin m \tilde{\phi} \end{array} \right)
\]

\[
C e_m(\tilde{u}, q) \rightarrow \text{const} \ J_m \left( \frac{\hbar a}{2} \exp \tilde{u} \right)
\]

so that finally

\[
\left( \begin{array}{c} c e_m(\tilde{\phi}, q) C e_m(\tilde{u}, q) \\ s e_m(\tilde{\phi}, q) S e_m(\tilde{u}, q) \end{array} \right)
\]

\[
\rightarrow \text{const} \ J_m \left( \frac{\hbar a}{2} \exp \tilde{u} \right) \left( \begin{array}{c} \cos m \tilde{\phi} \\ \sin m \tilde{\phi} \end{array} \right)
\]

so, as expected, the wavefunction (24) tends to the solution corresponding to the circular cylindrical system.

Considering other limiting cases, one can readily show that representations of the type (24) using
the radial functions $F_{k_m}$ and $G_{k_m}$, tends to the circular-cylindrical solutions containing the Neumann functions $N_{k_m}$, while the functions $F_{k_m}$ and $G_{k_m}$ are connected with the Hankel functions.

References


