1 Introduction

The overwhelming majority of theoretical studies on the wave motion rely upon the spectral-modal approach implying complete separation of variables in the descriptive equations by suitable integral transforms or series expansions in terms of eigenfunctions (modes). Of them, the most prominent is the Fourier (Laplace) transform lying in the basis of the frequency domain ansatz, whose nearly universal applicability disseminated a belief that a wave is an object having a phase and amplitude (or, at least, a set of such objects) – rather than a solution of the wave equation. Direct description of the processes in the space-time domain results in more complex and less universal ansatz, which however may turn out to be more adequate for solving specific problems, especially in the situations where the source term has a complicated structure or the boundness of the source and, as a consequence, the emanated wave (signal) is important. Notably, within the framework of the frequency domain ansatz a finite wave is represented by infinite number of components having unbounded support in space-time. For the wave motion at the speed of light the direct methods also permit avoiding some of the controversial issues connected with causality [1].

Construction of novel solutions to the homogeneous wave and Maxwell’s equations (in particular, the undistorted progressing waves) via the four-dimensional conformal Bateman transformations [2] is among the most straightforward methods [3]. For the inhomogeneous equations one can follow a well-established method based on the Green’s function technique [4]. This technique may be applied directly to the wave equation (an example related to localized waves is given in [5]) and for the circumstances described in Section 6.4 and Chapter 14 of [4] be reduced to calculation of the wave field via retarded (in particular, the Liénard-Wiechert) potentials [6].

Following another way, one can first separate one or two coordinates and apply Green’s method for solving the resulting hyperbolic PDE in fewer dimensions [7]. Such a technique of incomplete separation of variables has found extensive application in the theory of guided and free wave motion – especially in the case of the (general-type) cylindrical coordinates when the wave equation is reduced to the Klein-Gordon equation (KGE) whose solutions are still invariant under Lorentz transforms and subjected to the causality principle [8]. This chapter presents a brief overview of an alternative method developed within the scope of the last technique, based on constructing solutions of the KGE using the Riemann-Volterra approach. Although being specific and, in certain occasions, tedious, this method appears to be the most powerful within the scope of its applicability, yielding physically feasible and realizable solutions in the form of bounded-support, finite-energy waves. The causality of the solution is provided by minimal axiomatics invoked only at the initial stage – the problem posing, which includes time-asymmetric homogeneous initial conditions imposed upon the wavefunction and the source, admitting the existence of a moment (taken as \( t = 0 \)) prior to which neither the source nor the emanated wave can exist.

Despite certain similarity between Green’s and Riemann-Volterra methods (in some literature the Riemann function is called the Riemann-Green function [9]), their application to the problems of wave motion results in completely distinct situations:

(i) The definitions of both Green’s function and the corresponding Green’s solution are not unique as they leave room for addition of arbitrary solution of the homogeneous equation; in some circumstances the particular choice of Green’s function and the final solution are defined by bound-
ary condition(s) or plausibility and physical admissibility of the constructed wavefunctions (a comprehensive discussion of this subject can found in [10]). The Riemann function is a solution of the homogeneous equation that additionally must take a certain value at the characteristics and thus is defined in a unique way.

(ii). In contrast to Green’s method that provides a particular solution of the inhomogeneous equation, the Riemann-Volterra method is related to the corresponding problem, comprising the PDE and initial conditions [11, 12]; and it was the Riemann-Volterra representation that Smirnov used in his “Course of Higher Mathematics” to prove the uniqueness of the solution to the above problem (see [12], item 143).

(iii). In the general case, Green’s formula implies integration over the entire domain of variation of coordinates and time, while integration in the Riemann-Volterra solution is carried out within a limited triangle region, assuring, as we will show, the boundness of the solution support.

(iv). Causality of the (unique) Riemann-Volterra solution is provided automatically, without need to recur to additional considerations, such as the retarded nature of the argument, wave propagation in certain direction, specific choice of the integration path, etc.

(v). For traveling sources, the Green’s function can be readily derived from the Liénard-Wiechert potential of a moving point source. However, concrete calculation of the wavefunction, inevitably involving the analysis of the retarded argument, may develop in a rather complicated task unless some special techniques, like the parametric method [13], are invoked. The Riemann-Volterra approach presents the same or even more serious difficulties, especially when one deals with the bounded-support sources: here the actual limits of integration must be defined from the system of inequalities involving the space-time variables and parameters of the source term. However, this definition can be strictly formalized using 2D diagrams on the plane spanned by the time variable and unseparated coordinate, see [14] and references therein.

The relation between the heuristic methods of description of the signal propagation based on geometrical considerations (see, e.g., [15]) and the Riemann-Volterra approach resembles the relation between arithmetic and algebra: the former may rapidly give a straightforward ad hoc results while the latter paves a way for more general and formal consideration that automatically provide predictions of the simpler treatment a particular result.

The main objective of this chapter is to illustrate how such formal consideration can be tailored for the description of localized waves. As the Riemann-Volterra approach is suitable for sources with various spatiotemporal structures subjected to any type of motion, two distinctive examples of application, linked with the most prominent localized waves — Brittingham’s focus wave modes (FWM) [16] and X-shaped waves [17, 18, 19] — will be given. The former problem involves a partially (longitudinally) bounded source with the Gaussian transverse distribution travelling with the speed of light (sound), while the latter concerns a line source pulse propagating with a superluminal (supersonic) velocity. These particular applications are discussed in Sections 3 and 4, preceded by Section 2 devoted to the basics of the approach. Conclusive remarks are made in final Section 5.

2 Basics of the Riemann-Volterra approach

2.1 Problem posing

Let us consider a scalar wave problem in axisymmetric geometry comprising the wave equation

\[ \left[ \partial^2_\tau - \partial^2_z - \rho^{-1} \partial_\rho \left( \rho \partial_\rho \right) \right] \psi (\rho, z, \tau) = j (\rho, z, \tau) \]  \( (1) \)

and the initial condition

\[ \psi \equiv 0, \ j \equiv 0 \ \text{for} \ \tau < 0 \]  \( (2) \)

Here \( \tau = ct \) is time \( t \) in units of length reckoned from the moment of source excitation \( (\tau = 0) \), \( c \) the wavefront velocity, \( \rho, \varphi, z \) the circular cylindrical coordinate system \( (\partial_\varphi \equiv 0 \text{ due to the axial symmetry}) \), while \( \psi \) and \( j \) represent the wavefunction and the source term.
2.2 Riemann-Volterra solution

Being native for acoustics, in the case of electromagnetics such a problem can be obtained for the third components of the four-potential \( A^{(a)} = (A^{(0)}, 0, 0, \psi) \) and four-current \( J^{(a)} = (J^{(0)}, 0, 0, j) \) [20, 21]. Taking into account condition (2) and the Lorentz gauge \( \sum \alpha \partial_\alpha A^{(a)} = 0 \), one can find the other pair of non-zero components of the four-vectors (and thus get the complete description of the electromagnetic field)

\[
\begin{pmatrix}
A^{(0)} \\
J^{(0)}
\end{pmatrix} = - \int_0^\infty d\tau' \partial_3 \begin{pmatrix}
A^{(3)} \\
J^{(3)}
\end{pmatrix} = - \int_0^\infty d\tau' \partial_z \begin{pmatrix}
\psi \\
j
\end{pmatrix}
\]

(3)

which, due to the continuity equation \( \sum_\alpha \partial_\alpha J^{(a)} = 0 \), automatically satisfies (1), (2).

Performing the Fourier-Bessel transform

\[
\begin{pmatrix}
\tilde{\psi}(s, z, \tau) \\
\tilde{j}(s, z, \tau)
\end{pmatrix} = \int_0^\infty d\rho \rho \begin{pmatrix}
\psi(\rho, z, \tau) \\
j(\rho, z, \tau)
\end{pmatrix} J_0(s\rho)
\]

(4)

(\( J_0(\cdot) \) is the Bessel function of the first kind of order zero), we get from (1), (2) a simpler problem

\[
(\partial^2_\tau - \partial^2_z + s^2) \tilde{\psi}(s, z, \tau) = \tilde{j}(s, z, \tau), \quad \tilde{\psi} \equiv 0, \tilde{j} \equiv 0 \quad \text{for} \quad \tau < 0.
\]

(5)

2.2 Riemann-Volterra solution

Rotating the \( z, \tau \) plane clockwise through an angle \( \theta = \pi/4 \) about its origin and scaling it by a factor of \( 1/\sqrt{2} \), i.e., passing to the new variables

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = \frac{1}{\sqrt{2}} \mathbf{R}(\theta)|_{\theta=-\pi/4} \begin{pmatrix}
z \\
\tau
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} (z + \tau) \\
\frac{1}{\sqrt{2}} (-z + \tau)
\end{pmatrix}
\]

(6)

we arrive at the initial value problem for the KGE in its first canonical form

\[
\partial_X \partial_Y \tilde{\psi} = \tilde{j}, \quad \tilde{\psi} \equiv 0, \tilde{j} \equiv 0 \quad \text{for} \quad X + Y < 0,
\]

(7)

whose known Riemann function, \( R = J_0 \left( s \sqrt{4(X-X') (Y-Y')} \right) \) enables the unique solution to be constructed in the half-plane \( \Omega_1 = \{ X, Y : X + Y > 0 \} \), depicted in Fig. 1, using the Riemann-Volterra method (see [14], Section 3.2 for details)

\[
\tilde{\psi} = \iint_{\Delta_{QPM}} dX' dY' R \tilde{j}(s, X', Y').
\]

(8)

In the other half-plane \( \Omega_0 = \{ X, Y : X + Y < 0 \} \) the (zero) solution is uniquely defined by the initial conditions. Coming back to the variables

\[
z = \frac{1}{2} (X - Y), \quad \tau = \frac{1}{2} (X + Y) \quad \Rightarrow \quad \partial(X, Y) = \frac{1}{2},
\]

(9)

we have

\[
\tilde{\psi}(s, z, \tau) = \frac{1}{2} \int_{z-\tau}^{z+\tau} dz' \int_0^{\tau-|z-z'|} d\tau' R(s, z, \tau; z', \tau') \tilde{j}(s, z', \tau'),
\]

(10)

where the Riemann function takes the form

\[
R(s, z, \tau; z', \tau') = J_0 \left( s \sqrt{(\tau - \tau')^2 - (z - z')^2} \right).
\]

Eventually, the unique solution for the original problem (1), (2) can be found using the inverse Fourier-Bessel transform

\[
\psi(\rho, z, \tau) = \int_0^\infty ds s \tilde{\psi}(s, z, \tau) J_0(s\rho) = \frac{1}{2} \int_{z-\tau}^{z+\tau} dz' \int_0^{\tau-|z-z'|} d\tau' J_0 \left( s \sqrt{(\tau - \tau')^2 - (z - z')^2} \right) \tilde{j}(s, z', \tau'),
\]

(11)

function \( \tilde{j}(s, z', \tau') \) being defined via \( j(\rho, z', \tau') \) by formula (4).
3 Emanation from wavefront-speed source pulse of Gaussian transverse variation: Causal clipped Brittingham’s focus wave mode

Let us take the source in the form of a pulse with the Gaussian transverse and Dirac longitudinal distribution propagating with the wavefront velocity $c$ along the $z$ axis

$$j (\rho, z, \tau) = \delta (\tau - z) F(z, \tau) \left[ \frac{1}{\pi a^2} \exp \left( -\rho^2 / a^2 \right) \right] \quad \text{for} \quad \tau > 0,$$

where the function $F(z, \tau)$ describes the pulse shape. As far as $\delta (\tau - z) F(z, \tau) = \delta (\tau - z) F(\tau, \tau)$, the model admits description of the pulse shape via a one-argument function. In particular, we can choose

$$F(z, \tau) = f \left( \frac{1}{2} (\tau + z) \right),$$

which without loss of generality transforms (12) into

$$j (\rho, z, \tau) = \delta (\tau - z) f \left( \frac{1}{2} (\tau + z) \right) \left[ \frac{1}{\pi a^2} \exp \left( -\rho^2 / a^2 \right) \right] \quad \text{for} \quad \tau > 0.$$

Calculation of

$$\tilde{j}(s, z, \tau) = \frac{1}{\pi a^2} \delta (\tau - z) f \left( \frac{1}{2} (\tau + z) \right) \int_0^\infty dp \rho \exp \left( -\frac{\rho^2}{a^2} \right) J_0(s \rho)$$

$$= \frac{1}{2\pi} \delta (\tau - z) f \left( \frac{1}{2} (\tau + z) \right) \exp \left( -\frac{a^2 s^2}{4} \right)$$

is facilitated by relation (6.631-4) of Gradshteyn and Ryzhik [22]

$$\int_0^\infty dx x \exp \left( -\alpha x^2 \right) J_0(\beta x) = \frac{1}{2\alpha} \exp \left( -\frac{\beta^2}{4\alpha} \right)$$

and brings us to the following concretization of general solution (10)

$$\psi (\rho, z, \tau) = \frac{1}{4\pi} \int_{z-\tau}^{z+\tau} ds' \int_0^{\tau - |z - z'|} d\tau' \delta (\tau' - z') f \left( \frac{1}{2} (\tau' + z') \right)$$

$$\times \int_0^\infty ds J_0(s \rho) J_0 \left( s \sqrt{\frac{(\tau - \tau')^2 - (z - z')^2}{4}} \right) \exp \left( -\frac{a^2 s^2}{4} \right).$$

On the $X', Y'$ plane of Fig. 1 the wavefunction $\psi (\rho, z, \tau)$ takes the form

$$\psi (\rho, X - Y, X + Y) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dY' \int_{-\infty}^{\infty} dX' \frac{\partial}{\partial (X', Y')} \delta (z', \tau') \left( 2Y \right)$$

$$\times f (X') \int_0^\infty ds J_0(s \rho) J_0 \left( s \sqrt{4(X - X')(Y - Y')} \right) \exp \left( -\frac{a^2 s^2}{4} \right).$$
Bearing in mind that \( \frac{\partial (x', \tau')}{\partial (X', \varphi')} = 2, \delta (2x') = \frac{1}{2} \delta (x') \) and \( \int_{-X}^{X} dY' \delta (x') \mathcal{F} (Y') = h (Y) h (X) \mathcal{F} (0), \) where \( h (\cdot) \) denotes the Heaviside (unit) step function, we can rewrite the wavefunction in a simpler form

\[
\psi = \frac{1}{4\pi} h (X) h (Y) \int_{0}^{X} dX' f (X')
\times \int_{0}^{\infty} ds \ s \exp \left( -\frac{a^2 s^2}{4} \right) J_0 (s \rho) J_0 \left( s \sqrt{4 (X - X') Y} \right),
\]

whose further reduction is possible using formula (6.633-2) of [22]

\[
\int_{0}^{\infty} dx \exp (-\gamma^2 x^2) J_0 (\alpha x) J_0 (\beta x) = \frac{1}{2\gamma^2} \exp \left( -\frac{\alpha^2 + \beta^2}{4\gamma^2} \right) I_0 \left( \frac{\alpha \beta}{2\gamma^2} \right)
\]

where \( I_0 (x) = J_0 (ix) \) is the modified Bessel function of the first kind of order zero: the inner integral of (18) can be represented in an analytical form, reducing the solution to

\[
\psi = \frac{h (X) h (Y)}{2\pi a^2} \exp \left( -\frac{\rho^2 + 4Xy}{a^2} \right)
\times \int_{0}^{X} dX' f (X') \exp \left( \frac{4Xy}{a^2} \right) I_0 \left( \frac{4\rho \sqrt{(X - X') y}}{a^2} \right).
\]

Let us take \( f (X') = \text{Re} \left[ -\frac{1}{4\pi} \exp (i2kyX) \right], k \) being a real constant, thus representing the source (13) in the dimensionless form and the wavefunction in units of length squared. Then Eq. (19) yields

\[
\psi = \text{Re} \left[ -\frac{h (X) h (Y)}{2\pi a^2} \exp \left( -\frac{\rho^2 + 4Xy}{a^2} \right) \right]
\times \int_{0}^{X} dX' \frac{4\pi}{k^3} \exp \left( i2 \left( kX' - \frac{2X' Y}{a^2} \right) \right) I_0 \left( \frac{4\rho \sqrt{(X - X') y}}{a^2} \right)
\]

From this, it is straightforward to infer the relation

\[
\psi = \text{Re} \left[ -\frac{h (X) h (Y)}{k^3} \exp \left( -\frac{\rho^2 + 4Xy}{a^2} \right) \frac{1}{ka^2 - i2Y} I_{wu} \right],
\]

\[
w \overset{\text{def}}{=} 4X \left( k - i2 \frac{Y}{a^2} \right), \quad u \overset{\text{def}}{=} \frac{4\rho \sqrt{Xy}}{a^2},
\]

\[
I_{wu} \overset{\text{def}}{=} w \int_{0}^{1} d\xi \ \xi J_0 (u \xi) \exp \left( \frac{i}{2} w \left( 1 - \xi^2 \right) \right),
\]

where the substitution of the integration variable \( X' \rightarrow \xi = \sqrt{1 - X'/X} \) has been performed.

Now we can take advantage of Watson’s results (see [23], chapter XVI, in particular, Eq. (3) of item 16.3 and Eq. (5) of item 16.53) and describe the integral \( I_{wu} \) in terms of Lommel’s functions of two variables

\[
U_n (w, u) = \sum_{m=0}^{\infty} (-1)^m \left( \frac{w}{u} \right)^{n+2m} J_{n+2m} (u),
\]

\[
V_n (w, u) = \sum_{m=0}^{\infty} (-1)^m \left( \frac{w}{u} \right)^{n+2m} J_{-n-2m} (u),
\]

as follows

\[
I_{wu} = \begin{cases} 
U_1 (w, u) + iU_2 (w, u) & \text{if } |w| < |u| \\
V_1 (w, u) + iV_0 (w, u) - i \exp \left( \frac{i}{2} \left( w + \frac{u^2}{w} \right) \right) & \text{if } |u| < |w|.
\end{cases}
\]

5
Coming back to the $z, \tau$ representation, one has
\[\psi = \text{Re} \left[ -\frac{h(\tau - |z|)}{k^2} \frac{1}{ik (z - \tau) + (ka)^2} \exp \left( -\frac{\rho^2 + \tau^2 - z^2}{a^2} \right) I_{wu} \right], \tag{24}\]
\[w = 2(z + \tau) \left( k + i \frac{z - \tau}{a^2} \right), \quad u = i \frac{2\rho \sqrt{\tau^2 - z^2}}{a^2}. \tag{25}\]

Notably, the step function $h(\tau - |z|)$ appearing in the solution puts it in correspondence with the causality principle that prohibits a signal emanated from the plane $z = 0$ at $\tau = 0$ to manifest itself on any plane $z = \text{const}$ before the moment $\tau = |z|$.

Let us rewrite the last formula in a more explicit form
\[
\psi = \psi_{tr} + \psi_{FWM}, \quad \psi_{tr} = h(\tau - |z|) \text{Re} \left[ \left( \frac{1}{k^2} \frac{1}{ik (z - \tau) + (ka)^2} \exp \left( -\frac{\tau^2 + \rho^2 - z^2}{a^2} \right) \right) \right. \\
\left. \times \left\{ h(|u| - |w|) [U_1(w, u) + iU_2(w, u)] + h(|w| - |u|) [V_1(w, u) + iV_0(w, u)] \right\} \right], \tag{26}\]
\[
\psi_{FWM} = h(\tau - |z|) h(|w| - |u|) \times \text{Re} \left\{ \frac{i}{k^2} \frac{1}{ik (z - \tau) + (ka)^2} \exp \left( -\frac{\tau^2 + \rho^2 - z^2}{a^2} + \frac{i}{2} \left( \frac{w + u^2}{w} \right) \right) \right\}. \tag{27}\]

The first term of (26) represents a transient process while the second corresponds to a focus wave, whose space-time structure can be found using Eqs. (25)
\[
\frac{i}{2} \left( \frac{w + u^2}{w} \right) = -\frac{z^2 - \tau^2}{a^2} + \frac{1}{a^2} \frac{(kp)^2 (z - \tau)^2}{[k (z - \tau)]^2 + (ka)^4} \\
+ ik \left[ (z + \tau) + \frac{(kp)^2 (z - \tau)}{[k (z - \tau)]^2 + (ka)^4} \right], \tag{28}\]
\[
h(|w| - |u|) = h \left( |z + \tau| \left[ (z - \tau)^2 + k^2 a^4 \right] - \rho^2 |z - \tau| \right). \]

In the area of interest $\tau - |z| > 0$ the second step function takes the form
\[
h(|w| - |u|) = h \left( (z + \tau) \left[ (z - \tau)^2 + k^2 a^4 \right] + \rho^2 (z - \tau) \right), \tag{29}\]
and we eventually get
\[
\psi_{FWM} = h(\tau - |z|) h(\tau + \tau) \left( (z - \tau)^2 + k^2 a^4 \right) + \rho^2 (z - \tau) \left( \frac{ik (z + \tau) - \frac{(kp)^2}{ik (z - \tau) + (ka)^2}}{1} \right). \tag{30}\]

The obtained solution corresponds to potential of Brittingham’s transverse-electric focus wave mode of order zero (cf. Eq. (32) of [24] and Eq. (7) of [25]) whose support, however, is limited by the conditions $-\tau < z < \tau \Rightarrow \tau - z > 0, \tau + z > 0$ (reflecting the causality) and
\[
k (z + \tau) > -\frac{(kp)^2}{[k (z - \tau)]^2 + (ka)^4} k (z - \tau). \tag{31}\]

It is interesting to note that quite a different frequency-domain description of the focused wave modes [26] is also based on the Lommel’s function formalism. Evolution of $\psi_{FWM}$ with time is illustrated
in Fig. 2 using the longitudinal propagation variable \( \zeta = z - \tau \) and dimensionless parameters. The wavefunction demonstrates an oscillating behavior at the central front point \( \psi_{FWM}|_{\rho, \zeta = 0} = -k^{-4}a^{-2} \sin(2k\tau) \) due to “ripples” produced by \( \exp(ik(z + \tau)) \). The rest of the waveform (29) depends only on \( \zeta \) and corresponds to a steady clipped envelope propagating along the positive \( z \) direction with the wavefront velocity \( c \). As the time increases, the envelope expands in both \( \rho \) and \( -\zeta \) directions, tending the entire constructed wavefunction to the “pure” Brittingham’s FWM. Inequality (30) defines rather simple boundary of the wavefunction support in the dimensionless coordinates \( k\zeta, k(z + \tau) \), plotted in Fig. 3(a); the same support mapped into the conventional coordinates \( z, \tau \) is depicted in Fig. 3(b).

As seen from the plots, the component \( \psi_{FWM} \) always exists in the vicinity of the co-propagating edge of the emanated electromagnetic pulse, where \( (\tau - z)/\tau = \varepsilon_- \) can be treated as a small positive parameter. In this case

\[
|w| = 2(2 - \varepsilon_-)k\tau \sqrt{1 + \left(\frac{\varepsilon_- - \tau}{ka^2}\right)^2} = \mathcal{O}\left(\varepsilon_0\right),
\]

\[
|u| = 2\frac{\rho \tau}{a^2} \sqrt{\varepsilon_-(2 - \varepsilon_-)} = \mathcal{O}\left(\varepsilon_-^{1/2}\right),
\]

Figure 2: Evolution of \( \psi_{FWM} \) with time illustrated using the dimensionless parameters \( k\rho, k\zeta = k(z - \tau), k\tau \) and \( k^2\psi_{FWM}; ka = 1 \).
ensuring \(|w| > |u|\). In the vicinity of the counter-propagating edge, \((z + \tau) / \tau = \varepsilon_+ \ll 1\), one has

\[
|w| = 2\varepsilon_+ k \tau \sqrt{1 + \left[\frac{(2 - \varepsilon_+) \tau}{ka^2}\right]^2} = O(\varepsilon_+),
\]

\[
|u| = 2\frac{\rho\tau}{a^2} \sqrt{\varepsilon_+ (2 - \varepsilon_+)} = O(\varepsilon_+^{1/2}),
\]

resulting in \(|w| < |u|\). In the area \(k(\tau - z) \gg (ka)^2\) inequality (30) becomes

\[
\tau + z > \frac{\rho^2}{\tau - z} \quad \Rightarrow \quad \tau > \sqrt{\rho^2 + z^2},
\]

describing an expanding spherical pulse front centered at the spacetime point \(\rho = 0, z = 0, \tau = 0\) — as one might expect, bearing in mind that for the limiting case \(a \to 0\) the source (13) takes the form of a pointlike pulse moving with the wavefront velocity

\[
\lim_{a \to 0} j(\rho, z, \tau) = \delta(\tau - z) f \left(\frac{1}{2} (\tau + z)\right) \delta(\rho) 2\pi \rho > 0.
\]

Here, like in the case of \(\tau > |z|\) for the source distributed on the moving plane \(\tau - z = 0\), inequality (33) reflects (apparently automatically) the causality condition. Corresponding highly localized focus mode

\[
\psi_{FWM}|_{a=0} = \left. \frac{\rho^2 + z^2}{\tau - z} \right| \text{Re} \left\{ \frac{1}{k^3} \exp\left(ik \left[(\tau + z) - \frac{\rho^2}{\tau - z}\right]\right) \right\}
\]

is described in details in [27] (including the case of complexified constant \(k\)). Shrinking the source distribution into a point results in a pulsed wave that diverges at the central point of the leading edge \(\rho = 0, \zeta = 0\) and the transition to the steady-state wave is reduced to a jump discontinuity at \(\tau = \sqrt{\rho^2 + z^2}\). As the next section will show, similar behavior, characterized by larger area of divergence, holds in the case of a localized wave emanated by the superluminal pointlike source.
4 Emanation from a source pulse moving faster than the wavefront: Droplet-shaped waves

4.1 General solution for the superluminal (supersonic) motion

Let us now consider, in place of the Gaussian pulse (13), a line source pulse moving faster than the wavefront, whose arbitrary shape $f$ remains invariant during propagation. In electromagnetics, faster-than-light source currents may be associated with the motion of charged tachyon particles [20] or with superluminal patterns created by a coordinated motion of the subluminally moving constituents (one of such models discussed by Ziolkovski et al. [28] and later by Saari [29] involves superluminal sink-source-type fictitious currents [30]).

Assuming dealing with the electromagnetic problem described in Subsection 2.1 and bearing in mind the basic source current representation obtained in [20, 21], it is useful to state explicitly the proportionality of the source to the propagation velocity $\beta c$ (this factor is supposed to be constant and does not impair the generality of the discussion)

$$j (\rho, z, \tau) = \beta c h (z) h (\beta \tau - z) f (z - \beta \tau) \frac{\delta (\rho)}{2\pi \rho}, \quad \beta > 1. \quad (36)$$

As far as $h (z) h (\beta \tau - z) \equiv 0$ for negative $\tau$, $j (\rho, z, \tau) \equiv 0$ for $\tau < 0$, obeying initial condition (2). Concretization of KGE (5) reads

$$(\partial^2_\tau - \partial^2_z + s^2) \tilde{\psi} = \frac{\beta c}{2\pi} h (z) h (\beta \tau - z) f (z - \beta \tau) = h (z) h (-\zeta) f (\zeta),$$

where the propagation variable $\zeta$ now takes the form $\zeta = z - \beta \tau$ corresponding to one of the so-called V-cone variables [20]. The general Riemann-Volterra solution (10) gives at once

$$\tilde{\psi} (s, z, \tau) = \frac{\beta c}{4\pi} \int_{z-\tau}^{z+\tau} dz' \int_0^{\tau-|z-z'|} d\tau' J_0 \left( s \sqrt{(\tau - \tau')^2 - (z - z')^2} \right)$$

$$\times h (z') h (\beta \tau' - z') f (z' - \beta \tau'),$$

and after changing the order of integration the inverse Fourier-Bessel transform yields

$$\psi (\rho, z, \tau) = \frac{\beta c}{4\pi} \int_{z-\tau}^{z+\tau} dz' \int_0^{\tau-|z-z'|} d\tau' h (z') h (\beta \tau' - z') f (z' - \beta \tau')$$

$$\times \int_0^{\infty} ds s J_0 (sp) J_0 (s \rho') = \rho^{-1} \delta (\rho - \rho')$$

Using the closure equation (see, e.g., [31], p. 691) $\int_0^{\infty} ds s J_0 (sp) J_0 (s \rho') = \rho^{-1} \delta (\rho - \rho')$ and the representation of the delta function with simple zeros $\{\tau_i\}$ ([31], p. 87)

$$\delta (g (\tau)) = \sum_i \frac{\delta (\tau - \tau_i)}{|g (\tau_i)|}$$

(among the two simple zeros, only one lies within the integration limits), we get solution (39) in the form

$$\psi (\rho, z, \tau) = \frac{\beta c}{4\pi \rho} \int_{z-\tau}^{z+\tau} dz' \int_0^{\tau-|z-z'|} d\tau' h (z')$$

$$\times h (\beta \tau' - z') f (z' - \beta \tau') \delta (\tau - \tau_i + \sqrt{\rho^2 + (z-z')^2}) \frac{\sqrt{\rho^2 + (z-z')^2}}{\sqrt{\rho^2 + (z-z')^2}} \sqrt{\rho^2 + (z-z')^2}.$$
The following text provides a detailed explanation of the process and its implications:

- The integration path for the wavefunction (41) is shown in Figure 4.

- The half-plane $H_\beta = \{\beta \tau' - z' > 0\}$ is the support of the step function $h(\beta \tau' - z')$; the half-plane $H_{z'} = \{z' > 0\}$ is the support of the step function $h(z')$; and the hyperbola $\Gamma = \{\tau' - \tau + \sqrt{\rho^2 + (z - z')^2} = 0\}$ is the support of the delta function $\delta(\tau' - \tau + \sqrt{\rho^2 + (z - z')^2})$.

- The $z', \tau'$ plane diagrams can be harnessed for revealing the explicit structure of the constructed solution via a sequence of case formulas, see Fig. 5:

- If $-\infty < z < z_c \Leftrightarrow \tau_t > r$

  $$\psi = \begin{cases} 0 & -\infty < \tau < r \\ \frac{\beta c}{4\pi} \int_0^{z_1} dz' \frac{f(z' - \beta(\tau - \tau'))}{\rho^2 + (z - z')^2} & r < \tau < \infty, \end{cases}$$  

  (42)

- Otherwise ($z_c < z < \infty \Leftrightarrow \tau_t < r$)

  $$\psi = \begin{cases} 0 & -\infty < \tau < \tau_t \\ \frac{\beta c}{4\pi} \int_0^{z_1} dz' \frac{f(z' - \beta(\tau - \tau'))}{\rho^2 + (z - z')^2} & \tau_t < \tau < \infty, \end{cases}$$

  (43)

  where $z_{1,2} = \beta \gamma^2 \left[ \beta z - \tau \pm \sqrt{(\beta \tau - z)^2 - (\beta^2 - 1) \rho^2} \right]$, $z_c = \gamma \rho$, $\gamma = (\beta^2 - 1)^{-1/2}$, $\tau_t = (z + \rho/\gamma) / \beta$, $r = \sqrt{\rho^2 + z'^2}$, and $r' = \sqrt{\rho^2 + (z - z')^2}$.

Although the method does not resort to a priori causality conditions, the space-time structure of the solution admits easy posterior interpretation in terms of causal propagation of information. For instance, for an observation point $\rho, z$ there is no intersection between $H_\beta$ and $\Gamma$ ($\psi \equiv 0$) until the moment $\tau = \tau_t$, in which the hyperbola touches the boundary of $H_\beta$ in the tangency point $\tau'_t = (z - \gamma \rho) / \beta$, $z'_t = z - \gamma \rho$, as depicted in Fig. 5(a). For $z > z_c$ the value of $\tau_t = \tau_{SL} + \tau_L$ represents the minimum time necessary for the electromagnetic energy, originated in the space-time point $\tau' = \rho' = z' = 0$, to reach the observation point $\rho = \rho'$, $z = z'$ at the superluminal speed $c_\beta$ with the source-pulse front and then during time $\tau_L = \sqrt{\rho^2 + (z - z'_t)^2} = \beta \gamma \rho$ out of this axis toward $\rho, z$ at the luminal speed with the front of the emanated electromagnetic wave (see the upper diagram of Fig. 5(a)).

For cases corresponding to Figs. 5(a,b), Eq. (43) characterizes quantitatively a source pulse that appears to the observer “suddenly growing out of a point” $z'_t, \tau'_t$ at $\tau = \tau_t$ and expanding in the...
opposite directions, just as was predicted by Gron and Ramanujam et al. [15] on the basis of purely geometrical consideration (see also Fig. 15 of [32] or Fig. 1 of [33]). Notably, apart from [15], other pioneering papers showed how a superluminal source, after having suddenly revealed in an “optical-boom” phase, may subsequently appear as a couple of objects receding one from the other, in particular, in astrophysical observations (so-called “superluminal expansions”). For a detailed discussion of these models see [34] and references therein.

Causal solution (43) demonstrates other phenomena discussed in the literature from the standpoint of the extended special relativity: as expound in [32], the superluminal motions, although observed being forward in time, can appear reversed in direction; as well, the duality between the source and detector makes possible the emission to be observed as absorption. Here the “post-boom” evolution of the source pulse is observed as forward and backward expansion. The latter stops as the pulse reaches its origin $z' = 0$ (Figs. 5(c,d)) and at this point the (reversed) generation process appears to be the pulse absorption. In the limiting case of a $\delta$-pulse the half-plane $H_2$ degenerates into a line and the segment of integration $\Gamma_1$ into points $A$ and $B$, representing two images of the same tachyonic perturbation. Source $A$ is eventually absorbed at $z' = 0$, abruptly diminishing the wave amplitude (as will be shown later, by half).

Aiming at construction of the propagation-invariant waves, it is worthwhile to express the wave-function in terms of the propagation variable $\zeta = z - \beta \tau$. Passing from $\rho, z, \tau$ to $\rho, \zeta, \tau$ transforms:

(i) the non-zero wave condition $\tau > \tau_t$ into $\rho < -\gamma \zeta$, a causal analog of the condition $\rho < \gamma |\zeta|$, reported in [20] for the steady-state electromagnetic field of a charged tachyon — in agreement with the predictions of the extended theory of special relativity (see [33], especially Fig. 4, as well as earlier works on superluminal Lorentz transformations [35]);

(ii) the case-limiting condition $\tau < r$ into $\tau_1 < \tau < \tau_2$, where

$$\tau_{1,2} (\rho, \zeta) = \tau - z_{2,1} \beta = \gamma^2 \left[ -\beta \zeta \mp \sqrt{\zeta^2 - (\beta^2 - 1) \rho^2} \right]$$

are two roots of the equation

$$\tau_{1,2} - r(\rho, \zeta, \tau_{1,2}) = 0, \quad r(\rho, \zeta, \tau) = \sqrt{\rho^2 + (\zeta + \beta \tau)^2}$$

Figure 5: Instantiation of general solution for the observation point $\rho, z$; case $z > z_c$. 
(iii) case formulas (42), (43) into

Case A: \((-\infty < \zeta < -\rho/\gamma \text{ and } -\infty < \tau < \tau_1)\) or \(-\rho/\gamma < \zeta < \infty\)

\[\psi(\rho, \zeta, \tau) = 0,\] (46)

Otherwise Case B: \(\tau_1 < \tau < \tau_2\)

\[\psi(\rho, \zeta, \tau) = \frac{\beta^2 c}{4\pi} \int_{\tau_1}^{\tau_2} d\tau' \frac{I(\beta|\tau(\rho, \zeta, \tau')-\tau'|)}{r(\rho, \zeta, \tau')}.\] (47)

Otherwise (Case C: \(\tau_2 < \tau < \infty\))

\[\psi(\rho, \zeta, \tau) = \frac{\beta^2 c}{4\pi} \int_{\tau_1}^{\tau_2} d\tau' \frac{I(\beta|\tau(\rho, \zeta, \tau')-\tau'|)}{r(\rho, \zeta, \tau')}.\] (48)

The \(\zeta, \tau\) plane map of the three areas differing in the solution representation is shown in Fig. 6. Contour plot, illustrating the wave structure via a characteristic space-time scale \(\lambda_0\) and the dimensionless quantities \(\tilde{\rho} = \rho/\lambda_0, \tilde{\zeta} = \zeta/\lambda_0, \tilde{\tau} = \tau/\lambda_0, \tilde{\tau}_{1,2} = \tau_{1,2}/\lambda_0\), is given in Fig. 7(a) \((\beta = \sqrt{2}, \text{ as in numerical illustrations of Ref. [20]})\). The emanated wave has an expanding droplet-shaped support, independent of the source shape \(f\) and defined exclusively by the parameters \(\rho, \zeta, \tau, \beta\). For each fixed moment of time \(\tilde{\tau}\) the solid isoline \(\tilde{\tau}_1(\tilde{\rho}, \tilde{\zeta}) = \tilde{\tau}\) defines the boundary of the wavefunction support (area \(\tilde{\tau}_1 < \tilde{\tau}\) corresponding to \(\psi \neq 0\)) while the dashed isoline \(\tilde{\tau}_2(\tilde{\rho}, \tilde{\zeta}) = \tilde{\tau}\) traces the boundary between cases B and C. Fig. 7(b) represents the structure of this droplet-shaped wave for \(f(\zeta) = \delta(\zeta)\), the (dimensionless) wavefunction \(\psi\) being \(\psi\) normalized by \(\frac{c}{2\pi\lambda_0}\).

4.2 Droplet-shaped waves as causal counterparts of the X-shaped waves

For the infinitely short source current pulse \(f(\zeta) = \delta(\zeta)\), which in the case discussed in [20] led to the model of X wave generation by a charged tachyon, one readily arrives at the solution that in the area \(\zeta < -\rho/\gamma \text{ and } \tau > \tau_1\) \((\psi \neq 0)\) reads

\[\psi = \begin{cases} \frac{\beta c}{4\pi} \int_{\tau_1}^{\tau_2} d\tau' \frac{\delta(\tau(\rho, \zeta, \tau')-\tau')}{r(\tau(\rho, \zeta, \tau'))} & \tau_1 < \tau < \tau_2 \\ \frac{\beta c}{4\pi} \int_{\tau_1}^{\tau_2} d\tau' \frac{\delta(\tau(\rho, \zeta, \tau')-\tau')}{r(\tau(\rho, \zeta, \tau'))} & \tau_2 < \tau < \infty. \end{cases}\] (49)
Figure 7: (a) Contour plots illustrating the shape of the wavefunction support (solid isolines $\tilde{\tau}_1 = \tilde{\tau}$) and the boundary between cases B and C (dashed isolines $\tilde{\tau}_2 = \tilde{\tau}$) for $\tilde{\tau} = 0.5, 1, 1.5, \text{ and } 2$. (b) A snapshot of $\tilde{\psi}$, clipped by $\tilde{\psi} = 10$ in the vicinity of its singularities, taken at $\tilde{\tau} = 1$; $\tilde{\zeta}_{1,2}$ are defined by the case delimiting conditions $\tilde{\tau}_{1,2}(\tilde{\rho} = 0, \tilde{\zeta}_{1,2}) - \tilde{\tau} = 0$.

The equation for the delta-function roots coincides with (45), so these roots, $\tau' = \tau_{1,2}$, are defined by formula (44). Using (40) and passing to the dimensionless parameters finally reduce (49) to the formula

$$\tilde{\psi}(\tilde{\rho}, \tilde{\zeta}, \tilde{\tau}) = \begin{cases} \frac{1}{2} \frac{\beta}{\sqrt{\zeta^2 - (\beta^2 - 1)\rho^2}} & \tilde{\tau}_1 < \tilde{\tau} < \tilde{\tau}_2 \\ \frac{3}{2} & \tilde{\tau}_2 < \tilde{\tau} < \infty. \end{cases}$$ (50)

The second case of (50) corresponds to the solution of the inhomogeneous wave equation of Ref. [20], describing the steady-state X wave produced by a charged tachyon (delta-pulse source). Introduction of the initial moment of particle generation results in launching of the same propagation-invariant waveform, which is devoid of the advanced component and restricted by the droplet-shaped support illustrated in Fig. 7(a). One observes wavefunction behavior similar to that discussed in the previous section for $\psi_{FW,M}$: as the time increases, the waveform expands in both $\tilde{\rho}$ and $-\tilde{\zeta}$ directions, tending to the classical result—here, to the retarded part of the charged-tachyon X-shaped field, as illustrated in Fig. 8. For $\tilde{\tau}_1 < \tilde{\tau} < \tilde{\tau}_2$ the singularity corresponding to $\tilde{\tau}_2$ (in Fig. 5, to $z_1$) resides outside the integration segment $\Gamma_i$, diminishing $\psi$ by half.

5 Conclusive remarks

For electromagnetic problems reducible to the scalar problem discussed in the previous section, the field vectors, of which only $E_\rho$, $E_z$, and $B_\phi$ components are nonzero, can readily be found from the obtained solution with the help of formula (3). In particular, for the observation times $\tilde{\tau} > \tilde{\tau}_2$ (a steady-state wave zone located behind the singularities arising from the potential discontinuities on the case delimiting boundaries of Fig. 6) the magnetic induction (normalized by $\psi_0 \lambda_0^{-1}$) is characterized by $\tilde{B}_\phi = -\partial \tilde{\psi}/\partial \tilde{\rho} = -\tilde{\rho} \beta (\beta^2 - 1) \left[\tilde{\zeta}^2 - (\beta^2 - 1) \tilde{\rho}^2\right]^{-3/2}$. As in the case described in [20], Sec. III, it remains the only component that does not vanish when $\beta \to \infty$, revealing a “magnetic monopole” behavior. As put forward in [32, 33, 35], the reference frame in which the
Figure 8: Dynamics of the droplet-shaped wave propagation (the points on the vertex and the cone surface, in which $\tilde{\psi}$ diverges, are omitted).

Particle velocity tends to infinity plays for tachyons the same role as the rest frame for ordinary particles (bradyons), and there exists a duality between subliminal electric charges and superluminal magnetic monopoles. So, for $\beta \to \infty$ one might expect the magnetic field to have a structure akin to that of the electric field of a charged particle. Notably, earlier applications of the proposed technique to the waves emanated by subluminal sources (see, for instance, [14, 36]) result in peak- or ball-like shapes akin to the subliminal wave bullets obtained in Secs. II-IV of Ref. [37]. While electromagnetic fields of different line currents propagating at luminal speed present singularity at one “point of accumulation”, $\rho = 0, \zeta \equiv z - \tau = 0$ (cf., e.g., wavefunction (35) of this study as well as that depicted in Fig. 1 of [27]), for the superluminal delta current the area of singularity spreads along the conical surface $\zeta = -\rho/\gamma$, representing, according to [32, 33] an initially point-like structure (perceived by superluminal observers) highly distorted by the superluminal Lorentz transformation. In toto, the results obtained support the general idea about the shape of superluminal particles and distribution of the field associated with superluminal charges: “while the simplest subluminal object is obviously a sphere or, in the limit, a space point, the simplest superluminal object is on the contrary an X-shaped pulse” [37].

Although the presented analysis is limited to the most illustrative cases of generation of the bounded-support analogs of the focus wave mode and X-shaped wave, the general integral solution (11) allows to investigate, both analytically and numerically, causal emanation of various other types of finite localized waves by pulsed sources. Various possible ansätze are discussed in [14] for free space and [38] for waveguides.
References


[8] O. A. Tretyakov and O. Akgun, Derivation of Klein-Gordon equation from Maxwell’s equations and study of relativistic time-domain waveguide modes, *Prog. Electromagn. Res.* **105**, 171–191 (2010). Separation of the time variable results in the Helmholtz equation, which is of elliptic type, and separation of the angular variables $\theta, \varphi$ in the spherical coordinate system $r, \theta, \varphi$ in the Euler-Poisson-Darboux equation, which is of hyperbolic type. In both cases the solution can be found using Green’s formula, but it is not invariant under Lorentz transforms.


